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RUDIMENTARY TREATISE

ON

PLANE AND SPHERICAL TRIGONOMETRY.

BY JAMES HANN,

OF KING'S COLLEGE.

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THE ELEMENTS

OF

PLANE TRIGONOMETRY.

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ADVERTISEMENT TO THE SECOND EDITION.

MR. HANN'S Rudimentary Treatise on Plane Trigonometry having been carefully revised and corrected, the publisher submits this new edition to the judgment of Mathematical Teachers and Students with an increased amount of confidence. He would also respectfully invite their attention to his cheap collection of Mathematical Tables, forming Vols. 94 and 95 of his Series of Scientific Treatises; they will be found amply sufficient for all the practical purposes of Trigonometry, and will therefore furnish every necessary aid in computations connected with the present subject. The following brief extract, from the Author's Preface to the former edition, is perhaps sufficiently descriptive of the character of the work:

"I have given, to illustrate the principles, a great number of examples fully worked out, and which I hope will be of service to those who have not the aid of a teacher.

"In compiling the work, the best authors, whether French or English, have been consulted. I may refer to the excellent Works of Bonnycastle, Cape, De Morgan, Gaskin, Hall, Hind, Hymers, Snowball, Woodhouse, and Gregory; and to Davies's edition of Hutton's Course.

"The problems have been principally taken from the Ladies and Gentleman's Diaries, the Cambridge Problems, and Leybourn's Repository.

"The demonstration of Demoivre's Theorem is taken from an able French work on Trigonometry, by Lefebure De Fourcy."

J. HANN.

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TRIGONOMETRY.

CHAPTER I.

1. TRIGONOMETRY was originally considered to be the doctrine of triangles, but in its present improved state it has a much more extensive signification, which we shall hereafter shew even in this rudimentary treatise.

2. In estimating angular measures, we suppose the right angle to be the primary one, and to be divided into 90 equal parts, each of which is called a degree; each degree is supposed to be divided into 60 equal parts, each of which is called a minute; each minute is supposed to be divided into 60 equal parts, each of which is called a second, and so on to thirds, fourths, &c. Here one degree is considered as the angular unit.

3. Modern French writers, instead of using the sexagesimal division, use the centesimal; and it is to be regretted that the latter is not universally used, from the great ease with which all calculations are made in that division.

We shall, however, shew how to reduce French into English measures, and *vice versâ*.

If E and F represent the number of English degrees and French grades in the same angle,

$$\frac{E}{90} = \frac{F}{100} \text{ or } \frac{E}{9} = \frac{F}{10};$$

$$\therefore E = \frac{9F}{10} = F - \frac{F}{10}; \text{ and } F = \frac{10E}{9} = E + \frac{E}{9}.$$

4. The circumference of a circle is known to be about 3·14159 times its diameter, or, in other words, the ratio of the circumference to the diameter is represented by 3·14159; for this number writers generally put the Greek letter π.

$\therefore$ circumference $= \pi D$; where D is the diameter, or $2\pi r$, where r is the radius of the circle

Hence the length of the arc of a quadrant is $\frac{\pi r}{2}$; of a semi-circle, or 180^0, is πr; and of 270^0, or three quadrants, is $\frac{3\pi r}{2}$.

Now if any arc a subtend an angle of A^0, then since $\frac{\pi r}{2}$ subtends 90^0, and that by Euclid VI. 33, angles are proportional to the arcs which subtend them,

$$A^0 : 90^0 :: a : \frac{\pi r}{2};$$

$$\therefore\ A^0 = \frac{180^0}{\pi} \cdot \frac{a}{r} \ldots\ldots\ldots\ldots (1).$$

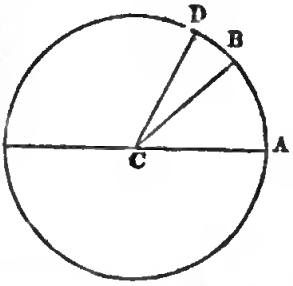
From this expression any one of the quantities may be found when the others are given.

Ex. 1. Find the length of an arc of 45^0 of a circle whose radius is 10 feet;

$$45^0 = \frac{180^0}{\pi} \cdot \frac{a}{10};$$

$$\therefore\ a = \frac{45^0}{180^0} \times 10 \times 3{\cdot}14159 = 7{\cdot}8539 \text{ feet.}$$

5. Most modern writers on Trigonometry take also for the unit of angular measure the number of degrees in an angle, subtended by an arc equal to the radius*. If U^0 represent that angle, then by equation (1),



* If ACB be an angle at the centre of a circle, subtended by an arc equal to the radius of the circle, then, since by the 33rd Proposition of the 6th Book of Euclid, the angles at the centre of a circle are to each other as the arcs on which they stand,

Angle ACB : four right angles :: arc AB : circumference, but AB is an arc equal to the radius,

$\therefore$ Angle ACB : four right angles :: $r : 2\pi r :: 1 : 2\pi$,

$\therefore\ ACB = \frac{\text{four right angles}}{2\pi}$, which, being independent of r, is constant for any circle; it may therefore be used to measure other angles.

$$U^0 = \frac{180^0}{\pi} \cdot \frac{r}{r} = \frac{180^0}{\pi} = \frac{180^0}{3{\cdot}14159} = 57^0{\cdot}29578.$$

Hence, $A^0 = 57^0{\cdot}29578\left(\frac{a}{r}\right)$ or $A^0 = U^0\left(\frac{a}{r}\right)$ ..... (2).

And since $U^0 = 57^0{\cdot}29578$ is constant,

$$A^0 \text{ varies as } \frac{a}{r}, \text{ } i.e. \text{ as } \frac{\text{arc}}{\text{radius}};$$

and taking U^0 as the unit, we have

$A^0 = \frac{a}{r}$ which is called the circular measure of the angle.

From equation (2) we see that the measuring unit, U^0, must be multiplied by the fraction $\frac{a}{r}$ to find the angle; thus if the circular measure of an angle be $\frac{5}{10}$, then

$$A^0 = \frac{5}{10}(57^0{\cdot}29578) = 28^0\ 64789.$$

If the circular measure be $\frac{6}{5}$ then

$$A^0 = \frac{6}{5}(57^0{\cdot}29578) = \frac{12}{10}(57^0{\cdot}29578) = 68^0{\cdot}754936.$$

Now, suppose we take an angle of $22^0\ 27'\ 39''$, then this is put into decimals at once by the centesimal division, without putting down any work on paper, it being $22^0{\cdot}2739$; whereas, by the sexagesimal, we must proceed in the following manner:

$$\begin{array}{r|l} 60 & 39 \\ \hline 60 & 27{\cdot}65 \\ \hline & 22{\cdot}4608 \end{array}$$

If we wish to find how many grades and minutes are contained in this angle, here

$$E = 22{\cdot}4608$$

$$\frac{E}{9} = 2{\cdot}4956$$

$E + \frac{E}{9} = 24{\cdot}9564$, which at sight is $24^g\ 95'\ 64''$.

Find the number of degrees and minutes in $46^g\ 56'\ 36''$.

$$F = 46{\cdot}5636$$
$$\frac{F}{10} = 4{\cdot}65636$$
$$E = F - \frac{F}{10} = 41{\cdot}90724$$
$$60$$
$$54{\cdot}4344$$
$$60$$
$$26{\cdot}064$$
$$E^0 = 41^0\ 54'\ 26''.$$

(1) If F' and F'', E' and E'' represent the magnitude of a French and English minute and second respectively, shew that

$$\frac{F'}{E'} = \frac{3{\cdot}3^2}{2{\cdot}5^2} : \frac{F''}{E''} = \frac{3{\cdot}3^3}{2{\cdot}5^3}.$$

$F' = 1$ French min.;

$\therefore F' \times 100 \times 100 =$ a quadrant,

$E' = 1$ English min.;

$\therefore E' \times 60 \times 90 =$ a quadrant,

hence, $F' \times 10000 = E' \times 5400$,

$F' \times 50 = E' \times 27$,

$$\frac{F'}{E'} = \frac{27}{50} = \frac{3{\cdot}3^2}{2{\cdot}5^2}.$$

Also, $F'' = 1$ French second;

$\therefore F'' \times 100 \times 100 \times 100 =$ a quadrant,

$E'' = 1$ English second;

$\therefore E'' \times 60 \times 60 \times 90 =$ a quadrant,

hence, $F'' \times 1000000 = E'' \times 324000$,

$F'' \times 250 = E'' \times 81$;

$$\frac{F''}{E''} = \frac{81}{250} = \frac{3{\cdot}3^3}{2{\cdot}5^3}.$$

(2) Compare the interior angles of a regular octagon and dodecagon:

In a polygon of n sides, $A^0 = 180^0 - \frac{360^0}{n}$.

In octagon $= 180^0 - \frac{360^0}{8} = 180^0 - 45^0 = 135^0$.

$$\text{In dodecagon} = 180^\circ - \frac{360^\circ}{12} = 180^\circ - 30^\circ = 150^\circ,$$

$$135 : 150 :: 9 : 10.$$

The ratio is therefore 9 to 10.

(3) The earth being supposed a sphere, of which the diameter is 7980 miles, find the length of an arc of 1°.

$$A^\circ = \frac{180^\circ}{\pi} \times \frac{a}{r}; \quad \therefore\ 1^\circ = \frac{180^\circ}{\pi} \times \frac{a}{\frac{7980}{2}} = \frac{180^\circ}{\pi} \times \frac{a}{3990},$$

$$\text{but } \frac{180^\circ}{\pi} = 57^\circ{\cdot}29578;$$

$$\therefore\ 1^\circ = 57^\circ{\cdot}29578 \times \frac{a}{3990},$$

$$a = \frac{3990}{57{\cdot}29578} = 69{\cdot}6 \text{ miles.}$$

(4) Find the diameter of a globe when an arc of 25° of the meridian measures 4 feet.

$$A^\circ = \frac{180^\circ}{\pi} \times \frac{a}{r}; \quad \therefore\ 25^\circ = \frac{180^\circ}{\pi} \times \frac{4}{r} = \frac{180^\circ}{\pi} \times \frac{8}{2r};$$

$$\therefore\ 2r = \frac{180}{\pi} \times \frac{8}{25} = 57{\cdot}29578 \times \frac{8}{25}$$

$$= 18{\cdot}3346 \text{ feet.} \quad \text{Answer.}$$

(5) Find the number of degrees in a circular arc 30 feet in length, of which the radius is 25 feet.

$$A^\circ = \frac{180^\circ}{\pi} \times \frac{a}{r} = \frac{180^\circ}{\pi} \times \frac{30}{25} = \frac{180^\circ}{\pi} \times \frac{6}{5}$$

$$= 57^\circ{\cdot}29578 \times \frac{6}{5} = 68^\circ{\cdot}75493.$$

(6) Find the number of degrees in an angle of which the circular measure is ·7854, the value of π being 3·1416.

$$A^\circ = \frac{180^\circ}{\pi} \times \frac{a}{r} = \frac{180^\circ}{3{\cdot}1416} \times {\cdot}7854 = \frac{180^\circ}{4} = 45^\circ.$$

(7) The interior angles of a rectilineal figure are in arithmetical progression, the least angle is 120°, and the common difference 5°; required the number of sides.

Sum of angles $= n\pi - 2\pi$ by Euc. (1. 32), or

$120^0 + 125^0 + 130^0$, &c., to n terms $= 180^0 n - 360^0$,

$$\{240 + (n-1)5\}\frac{n}{2} = 180n - 360,$$

$$120n + \frac{5n^2 - 5n}{2} = 180n - 360,$$

$$240n + 5n^2 - 5n = 360n - 720,$$

$$5n^2 - 125n = -720,$$

$$n^2 - 25n + \frac{625}{4} = \frac{625}{4} - 144 = \frac{625 - 576}{4} = \frac{49}{4},$$

$$n - \frac{25}{2} = \pm\frac{7}{2},$$

$$n = \frac{25 \pm 7}{2} = 16 \text{ or } 9.$$

The last is the congruent value of n, since no angle can be so great as 180^0; $\therefore$ the figure has nine sides.

(8) One regular polygon has two sides more than another, and each of its angles exceeds each angle of the other polygon by 15^0; find the number of sides in each.

$$nA = n\pi - 2\pi, \text{ equation (1)},$$

$$(n-2)(A-15) = (n - 2\times 2)\pi,$$

$$nA - 2A - 15n + 30 = n\pi - 4\pi, \text{ equation (2)}.$$

Subtracting equation (2) from (1),

$$2A + 15n - 30 = 2\pi,$$

$$2A = 2\pi - 15n + 30,$$

$$A = \frac{2\pi - 15n + 30}{2}, \text{ and from equation (1)}$$

$$A = \frac{n\pi - 2\pi}{n} = \frac{180n - 360}{n};$$

$$\therefore \frac{2\pi - 15n + 30}{2} = \frac{180n - 360}{n},$$

$$390n - 15n^2 = 360n - 720,$$

$$15n^2 - 30n = 720,$$

$$n^2 - 2n = 48,$$

$$n^2 - 2n + 1 = 49,$$

$$n-1=\pm 7,$$
$$n=1\pm 7=8 \text{ or } -6,$$
$$\text{and } n-2=6.$$

An octagon and a hexagon.

(9) The angles in one regular polygon are twice as many as in another polygon; and an angle of the former is to an angle of the latter as 3 : 2; find the number of sides.

Let n = number of sides in 1st, A = each angle,
$2n$ = number of sides in 2nd, B = each angle,

$$A=\left(\frac{n-2}{n}\right)\pi;\quad B=\left(\frac{2n-2}{2n}\right)\pi=\frac{n-1}{n}\pi,$$
$$\frac{B}{A}=\frac{n-1}{n-2} \text{ also } =\frac{3}{2};$$
$$\therefore\ 2n-2=3n-6,$$
$$n=4,$$
$$2n=8.$$

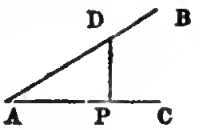
(10) The angles of a quadrilateral are in increasing geometrical progression, and the difference between the third angle and the fourth part of the first is 90^0; find the angles.

Let A, Ar, Ar^2, Ar^3, be the angles;

$$\therefore\ A(1+r+r^2+r^3)=360^0,$$
$$\text{and } A\left(r^2-\frac{1}{4}\right)=90^0;$$
$$\therefore\ (1+r+r^2+r^3)=(4r^2-1);$$
$$\therefore\ r^3-3r^2+r+2=0;$$
$$\therefore\ r^4-3r^3+r^2+2r=0,$$
$$(r^4-3r^3+\frac{9}{4}r^2)-(r^2-\frac{3}{2}r)+\frac{1}{4}=\frac{r^2}{4}-\frac{r}{2}+\frac{1}{4};$$
$$\therefore\ r^2-\frac{3}{2}r-\frac{1}{2}=\frac{r}{2}-\frac{1}{2};$$
$$\therefore\ r^2=2r;\quad \therefore\ r=2.$$
$$\text{But } A\left(r^2-\frac{1}{4}\right)=90^0;\quad \therefore\ \frac{15A}{4}=90^0$$
$$\therefore\ 15A=360^0;\quad \therefore\ A=24^0.$$

And the angles are 24^0, 48^0, 96^0, 192^0.

6. Trigonometrical Ratios or Definitions.



Let BAC be any angle, and from any point D in AB let fall the perpendicular DP on AC, then if we represent the angle BAC by A,

$\frac{DP}{AD}$ is the sine of A; $\frac{AP}{AD}$ is the cosine of A;

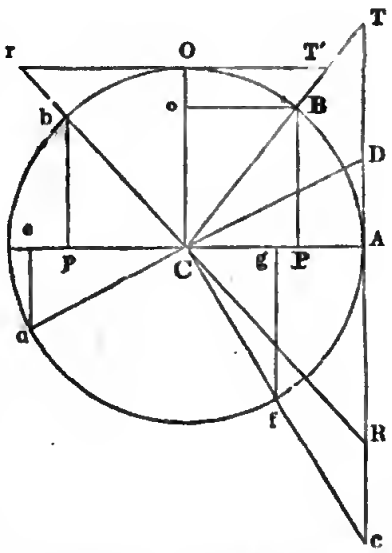
$\frac{DP}{AP}$ is the tangent of A; $\frac{AD}{DP}$ is the cosecant of A;

$\frac{AD}{AP}$ is the secant of A; $\frac{AP}{DP}$ is the cotangent of A.

For the sake of abbreviation the above quantities are generally put sin A, cos A, tan A, sec A, cosec A, cot A. $1 - \cos A$ is defined to be the versed sine of A, or vers A*.

By the 47th Prop. of the 1st Book of Euclid,

$$AP^2 + DP^2 = AD^2 \ldots\ldots\ldots\ldots\ldots (1)$$



* It may be perhaps necessary to remark that the following definitions have been given by most English writers till within the last few years. In the annexed figure, take any arc AB, draw BP and AT each perpendicular to the diameter AE, and produce CB to meet AT in T, then BP is called the sine of the angle ACB to the radius CB; CP is called the cosine; AT the tangent; CT the secant; AP the versed sine; OT' the cotangent; CT' the cosecant.

If we take the arc Ab greater than one quadrant and less than two, then bp is called the sine; Cp the cosine; AR the tangent; CR the secant; Ap the versed sine.

If the arc Abd be greater than two quadrants but less than three, then de is called the sine; Ce the cosine; AD the tangent; CD the secant; Ae the versed sine.

If the arc $Abdf$ be greater than three quadrants hut less than four, then fg is called the sine; Cg the cosine; Ac the tangent; Cc the secant; and Ag the versed sine.

Let the angle $ACB = A$, r = radius CB, then since $CP^2 + PB^2 = CB^2$ by 47th Prop. 1st book of Euclid, we have $\cos^2 A + \sin^2 A = r^2$ .....(1)

Also $AC^2 + AT^2 = CT^2$, or

$$r^2 + \tan^2 A = \sec^2 A \ldots\ldots\ldots\ldots\ldots (2)$$

Divide both sides of this equation by AD^2, and we have

$$\frac{AP^2}{AD^2} + \frac{DP^2}{AD^2} = \frac{AD^2}{AD^2};$$

$$i.e.\ \left(\frac{AP}{AD}\right)^2 + \left(\frac{DP}{AD}\right)^2 = 1:$$

but $\frac{AP}{AD}$ is the cosine of A, and $\frac{DP}{AD}$ is the sine of A, by the above definitions;

$$\therefore \cos^2 A + \sin^2 A = 1 \ldots\ldots\ldots\ldots\ldots (2).$$

From this equation we have,

$$\cos^2 A = 1 - \sin^2 A;$$

$$\therefore \cos A = \sqrt{(1 - \sin^2 A)},$$

$$\text{and } \sin^2 A = 1 - \cos^2 A,$$

$$\sin A = \sqrt{(1 - \cos^2 A)};$$

and since by the definitions $\frac{DP}{AD} = \sin A$, and $\frac{AP}{AD} = \cos A$, we have $DP = AD \sin A$, and $AP = AD \cos A$.

$$\text{Now } \tan A = \frac{DP}{AP} = \frac{AD \sin A}{AD \cos A} = \frac{\sin A}{\cos A},$$

From (1) $\cos^2 A = r^2 - \sin^2 A$, or $\cos A = \sqrt{(r^2 - \sin^2 A)}$

and $\sin^2 A = r^2 - \cos^2 A$, or $\sin A = \sqrt{(r^2 - \sin^2 A)}$

From (2) $\tan^2 A = \sec^2 A - r^2$.

The triangles BCP, TCA, OCT', and oCB, are all similar,

$$\therefore \frac{AT}{AC} = \frac{BP}{CP} \text{ or } \frac{\tan A}{r} = \frac{\sin A}{\cos A}$$

$$\text{or } \tan A = \frac{r \sin A}{\cos A} \ldots\ldots\ldots\ldots\ldots (3)$$

$$\frac{\cot A}{r} = \frac{\cos A}{\sin A}$$

$$\text{or } \cot A = \frac{r \cos A}{\sin A} \ldots\ldots\ldots\ldots\ldots (4)$$

$$\frac{\sec A}{r} = \frac{r}{\cos A} \text{ or } \sec A = \frac{r^2}{\cos A} \ldots\ldots\ldots\ldots\ldots (5)$$

$$\frac{\operatorname{cosec} A}{r} = \frac{r}{\sin A} \text{ or } \operatorname{cosec} A = \frac{r^2}{\sin A} \ldots\ldots\ldots\ldots\ldots (6)$$

$$\text{versin } A = r - \cos A \ldots\ldots\ldots\ldots\ldots (7).$$

$$\cot A = \frac{AP}{DP} = \frac{AD \cos A}{AD \sin A} = \frac{\cos A}{\sin A},$$

$$\sec A = \frac{AD}{AP} = \frac{AD}{AD \cos A} = \frac{1}{\cos A},$$

$$\operatorname{cosec} A = \frac{AD}{DP} = \frac{AD}{AD \sin A} = \frac{1}{\sin A},$$

$$\sec^2 A = \frac{AD^2}{AP^2} = \frac{AP^2 + DP^2}{AP^2} = 1 + \left(\frac{DP}{AP}\right)^2 = 1 + \tan^2 A;$$

or by dividing equation (1) by AP^2,

$$\left(\frac{AP}{AP}\right)^2 + \left(\frac{DP}{AP}\right)^2 = \left(\frac{AD}{AP}\right)^2,$$

$$\text{or } 1 + \tan^2 A = \sec^2 A;$$

$$\therefore \sec A = \sqrt{(1 + \tan^2 A)}.$$

Also dividing equation (1) by DP^2, we have

$$\left(\frac{AP}{DP}\right)^2 + \left(\frac{DP}{DP}\right)^2 = \left(\frac{AD}{DP}\right)^2 \text{ that is}$$

$$\cot^2 A + 1 = \operatorname{cosec}^2 A;$$

$$\therefore \operatorname{cosec} A = \sqrt{(1 + \cot^2 A)}.$$

(1) The sine of an angle is equal to the cosine of its complement.

$$\text{Since } \angle ADP = 90^\circ - A,$$

$$\text{and } \sin ADP = \frac{AP}{AD} = \cos A,$$

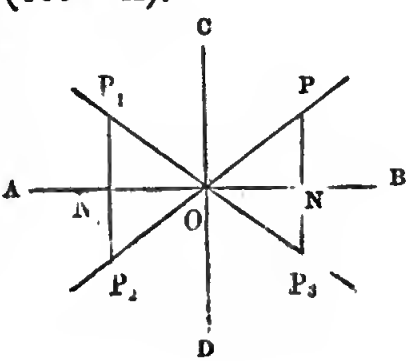
$$\cos ADP = \frac{DP}{AD} = \sin A;$$

$$\text{that is, } \sin (90^\circ - A) = \cos A,$$

$$\text{and } \cos (90^\circ - A) = \sin A.$$

(2) To shew that $\sin A = \sin (180^\circ - A)$.

Let BOP_1 be greater than a right angle : draw P_1N_1 perpendicular to AB; make the angle BOP equal to the angle AOP_1, and $OP = OP_1$, and let fall the perpendicular PN, then the triangles P_1ON_1 and PON are evidently equal.



$$\sin BOP_1 = \frac{P_1N_1}{OP_1};\ \sin BOP = \sin(180^\circ - BOP_1) = \frac{PN}{OP};$$

but $P_1N_1 = PN$, and $OP_1 = OP$;

$\therefore \sin BOP_1 = \sin(180^\circ - BOP_1)$:

that is, the sine of an angle is equal to the sine of the supplement of that angle:

$$\cos BOP_1 = \frac{ON_1}{OP_1} = -\frac{ON}{OP},$$

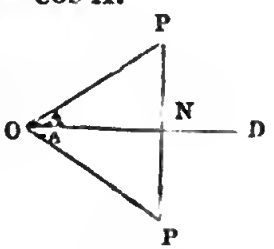
$$\cos BOP = \cos(180^\circ - BOP_1) = \frac{ON}{OP};$$

$$\therefore \cos BOP_1 = -\cos(180^\circ - BOP_1);$$

or the cosine of an angle is equal to minus the cosine of its supplement.

(3) $\sin(-A) = -\sin A$, and $\cos(-A) = \cos A$.

If the line OP revolve round the point O from OD upwards till it describes an angle $NOP = A$, then if it revolve downwards from OD till it describes the angle $NOP_1 = NOP$, then $NOP_1 = -A$.



$$\sin A = \frac{PN}{OP}, \text{ and } \sin NOP_1 = \sin(-A) = \frac{P_1N}{OP_1};$$

$$\text{but } \frac{P_1N}{OP_1} = -\frac{PN}{OP};$$

$$\therefore \sin(-A) = -\sin A;$$

$$\cos A = \frac{ON}{OP}, \text{ and } \cos NOP_1 = \cos(-A) = \frac{ON}{OP_1} = \frac{ON}{OP};$$

$$\therefore \cos A = \cos(-A).$$

From these we readily see that

$$\tan A = -\tan(-A),$$
$$\cot A = -\cot(-A),$$
$$\sec A = \sec(-A),$$
$$\operatorname{cosec} A = -\operatorname{cosec}(-A).$$

After one revolution is completed, the sines, cosines, &c., take the same values as before; therefore the sine of any angle is the same as the sine of 360° + that angle, or

$$\sin A = \sin(2\pi + A);$$

In the same way,

$$\sin A = \sin (4\pi + A),$$

and for the general form, where n is a whole number, we have

$$\sin A = \sin (2n\pi + A).$$

Similarly,

$$\sin (\pi - A) = \sin (3\pi - A) = \sin (5\pi - A);$$

or, generally,

$$\sin (\pi - A) = \sin \{(2n + 1)\pi - A\};$$

$$\text{but } \sin A = \sin (\pi - A);$$

$$\therefore \sin (2n\pi + A) = \sin \{(2n + 1)\pi - A\}.$$

$$\text{Also } \sin A = -\sin \{(2n + 1)\pi + A\}$$

$$= -\sin (2n\pi - A),$$

$$\cos A = \cos (2n\pi + A) = -\cos \{(2n + 1)\pi - A\},$$

$$\tan A = \tan (2n\pi + A) = -\tan \{(2n + 1)\pi - A\},$$

$$\sec A = \sec (2n\pi + A) = -\sec \{(2n + 1)\pi - A\}.$$

(4) In the figure at page 10, it is clear that if we suppose the line OP, originally coinciding with OB, to revolve round O as a centre, in each of the quadrants the various trigonometrical quantities will have precisely the same value; for the sake of distinction these quantities are affected with different signs. Thus, suppose $ON = ON_1 = x$, then ON, measured to the right of O, is called $+x$, and ON_1 measured to the left of O, is called $-x$. Also, any line PN above the line AB is considered positive, and any line P_3N below the same line is considered negative; thus if $PN = +y$, then P_3N is $-y$. This is purely conventional, for we might have taken lines to the left to be positive, and those to the right to be negative, and so on; but when we once fix on the positive direction, the negative direction must necessarily be opposite to it.

If we suppose the line OP to revolve upwards, the angle PON is considered positive; but, if downwards, the angle P_3ON is considered negative: thus, the angle PON is $+A$, and the angle P_3ON is $-A$.

(5) We may now proceed to trace the values of the sine, cosine, &c., throughout the four quadrants. In the first quadrant $\frac{PN}{OP}$, which is the sine of A, is positive; and $\frac{ON}{OP}$, which is the cosine of A, is also positive; it is clear that as

the sine increases, the cosine decreases, and when OP coincides with OC, then $PN = OP$, and $ON = 0$; hence the sin $90^0 = 1$ and cos $90^0 = 0$. In the second quadrant the sine is also positive, but the cosine or $\frac{ON_1}{OP_1}$ is negative, since ON_1 is measured in the opposite direction to ON; as OP revolves from OC towards OA the sine decreases, and the cosine increases negatively; so that when OP coincides with OA the sine is 0 and the cosine is -1, that is, sin $180^0 = 0$ and cos $180^0 = -1$.

In the third quadrant both the sine and cosine are negative, and when OP coincides with OD, then the sin $270^0 = -1$, and the cos $270^0 = 0$. In the fourth quadrant the sine is negative but the cosine positive, and when the line OP has completed a whole revolution by coinciding again with OB, then we have the

$$\text{sine } 360^0 = 0, \text{ and cosine } 360^0 = 1.$$

Since $\tan A = \frac{\sin A}{\cos A}$; in the first quadrant, both sin A and cos A being positive, tan A is also positive, and when $A = 90^0$ we have $\tan 90^0 = \frac{\sin 90^0}{0} = \frac{1}{0} = \text{infinity}$; that is, the tangent of 90^0 is infinite.

In the second quadrant $\tan A = \frac{+\sin A}{-\cos A}$, and is therefore negative, and $\tan 180^0 = \frac{\sin 180^0}{\cos 180^0} = \frac{0}{-1} = 0$.

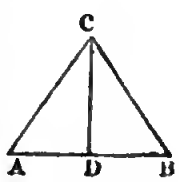
In the third quadrant $\tan A = \frac{-\sin A}{-\cos A}$, and is therefore positive, and $\tan 270^0 = \frac{\sin 270^0}{\cos 270^0} = \frac{-1}{0} = -\text{infinity}$.

In the fourth quadrant $\tan A = \frac{-\sin A}{+\cos A}$, and consequently negative, and $\tan 360^0 = \frac{\sin 360^0}{\cos 360^0} = \frac{0}{1} = 0$.

(6) To find sine, cosine, &c., of 30^0, 60^0, 45^0.

Let the angle be 60^0.

In the equilateral triangle ACB, let fall the perpendicular CD, which bisects both the angle ACB and the base AB.



Now, $\frac{AD}{AC} = \sin ACD = \sin 30^0$, but $AD = \frac{1}{2}AC$;

$$\therefore \sin 30^0 = \frac{\frac{1}{2}AC}{AC} = \frac{1}{2};$$

$$\therefore \cos 30^0 = \sqrt{(1 - \sin^2 30^0)} = \sqrt{\left(1 - \frac{1}{4}\right)} = \frac{\sqrt{3}}{2}, \text{ but}$$

$$\cos 30^0 = \sin 60^0; \ \therefore \sin 60^0 = \frac{\sqrt{3}}{2};$$

$$\cos 60^0 = \sin 30^0; \ \therefore \cos 60^0 = \frac{1}{2};$$

$\sin 30^0 = \frac{1}{2}$,	$\sin 60^0 = \frac{\sqrt{3}}{2}$,
$\cos 30^0 = \frac{\sqrt{3}}{2}$,	$\cos 60^0 = \frac{1}{2}$,
$\tan 30^0 = \frac{\sin 30^0}{\cos 30^0} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$,	$\tan 60^0 = \frac{\sin 60^0}{\cos 60^0} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$,
$\cot 30^0 = \frac{\cos 30^0}{\sin 30^0} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$,	$\cot 60^0 = \frac{\cos 60^0}{\sin 60^0} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$,
$\sec 30^0 = \frac{1}{\cos 30^0} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$,	$\sec 60^0 = \frac{1}{\cos 60^0} = \frac{1}{\frac{1}{2}} = 2$,
$\operatorname{cosec} 30^0 = \frac{1}{\sin 30^0} = \frac{1}{\frac{1}{2}} = 2$;	$\operatorname{cosec} 60^0 = \frac{1}{\sin 60^0} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$.

When the angle is 45^0,

$$\sin 45^0 = \cos (90^0 - 45^0) = \cos 45^0.$$

Now, since $\cos^2 45^0 + \sin^2 45^0 = 1$, and $\cos 45^0 = \sin 45^0$,

we have $2\cos^2 45^0 = 1$,

$$\cos^2 45^0 = \frac{1}{2},$$

$$\text{or, } \cos 45^0 = \frac{1}{\sqrt{2}};$$

$$\text{and therefore, } \sin 45^0 = \frac{1}{\sqrt{2}};$$

and since $\sin 45^0 = \cos 45^0$,

$$\tan 45^0 = \frac{\sin 45^0}{\cos 45^0} = 1,$$

$$\cot 45^0 = \frac{\cos 45^0}{\sin 45^0} = 1,$$

$$\sec 45^0 = \frac{1}{\cos 45^0} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2},$$

$$\operatorname{cosec} 45^0 = \frac{1}{\sin 45^0} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}.$$

The following results the student should commit to memory:

$$\cos^2 A + \sin^2 A = 1,$$

$$\cos A = \sqrt{(1 - \sin^2 A)}\,;\ \sin A = \sqrt{(1 - \cos^2 A)},$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{1}{\cot A} = \sqrt{(\sec^2 A - 1)},$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{1}{\tan A} = \sqrt{(\operatorname{cosec}^2 A - 1)},$$

$$\sec A = \frac{1}{\cos A} = \sqrt{(1 + \tan^2 A)},$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \sqrt{(1 + \cot^2 A)},$$

$$\operatorname{vers} A = 1 - \cos A.$$

Examples.

Ex. 1. If $\tan A = \frac{4}{3}$, compute the sine and versed sine.

$$\frac{\sin A}{\cos A} = \frac{4}{3},$$

$$3 \sin A = 4\sqrt{(1 - \sin^2 A)},$$

$$9 \sin^2 A = 16 - 16 \sin^2 A,$$

$$25 \sin^2 A = 16,$$

$$5 \sin A = 4,$$

$$\sin A = \frac{4}{5},$$

$$\cos A = \sqrt{(1 - \sin^2 A)} = \sqrt{\left(1 - \frac{16}{25}\right)} = \sqrt{\frac{9}{25}} = \frac{3}{5},$$

$$\text{vers } A = 1 - \cos A = 1 - \frac{3}{5} = \frac{2}{5}.$$

Ex. 2. $6 (\sin A)^2 = 5 \cos A$, find sine, cosine, tangent.

$$6 - 6 \cos^2 A = 5 \cos A,$$

$$6 \cos^2 A + 5 \cos A = 6,$$

$$\cos^2 A + \frac{5}{6} \cos A = 1,$$

$$\cos^2 A + \frac{5}{6} \cos A + \frac{25}{144} = 1 + \frac{25}{144} = \frac{169}{144},$$

$$\cos A + \frac{5}{12} = \pm \frac{13}{12},$$

$$\cos A = \frac{2}{3}, \text{ or } -\frac{3}{2},$$

$$\sin A = \sqrt{(1 - \cos^2 A)} = \sqrt{\left(1 - \frac{4}{9}\right)} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3},$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{\sqrt{5}}{3}}{\frac{2}{3}} = \frac{\sqrt{5}}{2}.$$

Ex. 3. $\sin A \cos A = \frac{\sqrt{3}}{4}$. Find sin, cos, and arc.

$$\sin A \sqrt{(1 - \sin^2 A)} = \frac{\sqrt{3}}{4},$$

$$\sin^2 A\,(1-\sin^2 A)=\frac{3}{16},$$

$$\sin^4 A-\sin^2 A=-\frac{3}{16},$$

$$\sin^4 A-\sin^2 A+\frac{1}{4}=\frac{1}{16},$$

$$\sin^2 A-\frac{1}{2}=\pm\frac{1}{4},$$

$$\sin^2 A=\frac{3}{4},\text{ or }\frac{1}{4},$$

$$\sin A=\frac{\sqrt{3}}{2},\text{ or }\frac{1}{2};$$

$$\therefore\ A=60^\circ,\text{ or }30^\circ.$$

$$\cos A=\frac{1}{2},\text{ or }\frac{\sqrt{3}}{2}.$$

Ex. 4. $\tan A+\cot A=\frac{4}{\sqrt{3}}$, find $\tan A$, and $\cot A$.

$$\tan A+\frac{1}{\tan A}=\frac{4}{\sqrt{3}},$$

$$\tan^2 A-\frac{4}{\sqrt{3}}\tan A=-1,$$

$$\tan^2 A-\frac{4}{\sqrt{3}}\tan A+\frac{4}{3}=\frac{1}{3},$$

$$\tan A-\frac{2}{\sqrt{3}}=\pm\frac{1}{\sqrt{3}},$$

$$\tan A=\sqrt{3},\text{ or }\frac{1}{\sqrt{3}},$$

$$\cot A=\frac{1}{\sqrt{3}},\text{ or }\sqrt{3}.$$

Ex. 5. If $3\sin A+5\sqrt{3}.\cos A=9$, find A.

$$5\sqrt{3}.\sqrt{(1-\sin^2 A)}=3\,(3-\sin A),$$

$$25.3.(1-\sin^2 A)=9\,(9-6\sin A+\sin^2 A),$$

$$25 - 25\sin^2 A = 27 - 18\sin A + 3\sin^2 A,$$

$$28\sin^2 A - 18\sin A = -2,$$

$$\sin^2 A - \frac{9}{14}\sin A = -\frac{1}{14},$$

$$\sin^2 A - \frac{9}{14}\sin A + \frac{81}{784} = \frac{81-56}{784} = \frac{25}{784},$$

$$\sin A - \frac{9}{28} = \frac{\pm 5}{28},$$

$$\sin A = \frac{\pm 5 + 9}{28} = \frac{1}{2}, \text{ or } \frac{1}{7};$$

$$\therefore\ A = 30^\circ.$$

Ex. 6. $\left.\begin{aligned}\sin A + \sin B &= \frac{1+\sqrt{3}}{2},\\ \sin A \sin B &= \frac{\sqrt{3}}{4}.\end{aligned}\right\}$ Find A and B;

$$\sin^2 A + 2\sin A \sin B + \sin^2 B = \frac{1 + 2\sqrt{3} + 3}{4},$$

$$4\sin A \sin B = \frac{4\sqrt{3}}{4},$$

$$\sin^2 A - 2\sin A \sin B + \sin^2 B = \frac{1 - 2\sqrt{3} + 3}{4},$$

$$\sin A - \sin B = \frac{\sqrt{3}-1}{2},$$

$$\sin A + \sin B = \frac{1+\sqrt{3}}{2},$$

$$2\sin A = 1; \quad \therefore\ \sin A = \frac{1}{2}; \quad \therefore\ A = 30^\circ,$$

$$2\sin B = \sqrt{3}; \quad \therefore\ \sin B = \frac{\sqrt{3}}{2}; \quad \therefore\ B = 60^\circ.$$

Ex. 7. Shew that $\sec^2 A \operatorname{cosec}^2 A = \sec^2 A + \operatorname{cosec}^2 A$.

$$\sec^2 A \operatorname{cosec}^2 A = \sec^2 A\,(1 + \cot^2 A) = \sec^2 A + \sec^2 A \cot^2 A,$$

$$= \sec^2 A + \frac{1}{\cos^2 A}\cdot\frac{\cos^2 A}{\sin^2 A} = \sec^2 A + \frac{1}{\sin^2 A}$$

$$= \sec^2 A + \operatorname{cosec}^2 A.$$

Ex. 8. If $\tan^2 A + 4\sin^2 A = 6$, prove that $A = 60°$.

$$\frac{\sin^2 A}{\cos^2 A} + 4\sin^2 A = 6, \text{ or, } \frac{\sin^2 A}{1-\sin^2 A} + 4\sin^2 A = 6,$$

$$\sin^2 A + 4\sin^2 A - 4\sin^4 A = 6 - 6\sin^2 A,$$

$$4\sin^4 A - 11\sin^2 A = -6,$$

$$\sin^4 A - \frac{11}{4}\sin^2 A + \frac{121}{64} = \frac{121}{64} - \frac{96}{64} = \frac{25}{64},$$

$$\sin^2 A - \frac{11}{8} = \pm\frac{5}{8}; \quad \therefore \sin^2 A = \frac{11}{8} \pm \frac{5}{8} = 2, \text{ or } \frac{3}{4};$$

$$\therefore \sin A = \frac{\sqrt{3}}{2} = \sin 60°; \quad \therefore A = 60°.$$

The $\sqrt{2}$ being greater than unity is inadmissible.

Ex. 9. If $\sin x \cos x + a\sin^2 x = b$, find x.

$$\sin x \sqrt{(1-\sin^2 x)} = b - a\sin^2 x,$$

$$\sin^2 x - \sin^4 x = b^2 - 2ab\sin^2 x + a^2\sin^4 x,$$

$$(a^2+1)\sin^4 x - (2ab+1)\sin^2 x = -b^2,$$

$$\sin^4 x - \left(\frac{2ab+1}{a^2+1}\right)\sin^2 x = -\frac{b^2}{a^2+1},$$

$$\sin^4 x - \left(\frac{2ab+1}{a^2+1}\right)\sin^2 x + \left\{\frac{2ab+1}{2(a^2+1)}\right\}^2$$

$$= \frac{4a^2b^2 + 4ab + 1}{4(a^2+1)^2} - \frac{b^2}{a^2+1}$$

$$= \frac{4ab + 1 - 4b^2}{4(a^2+1)^2},$$

$$\sin^2 x - \frac{2ab+1}{2(a^2+1)} = \pm\frac{\sqrt{(4ab+1-4b^2)}}{2(a^2+1)},$$

$$\sin^2 x = \frac{2ab+1 \pm \sqrt{(1+4ab-4b^2)}}{2(a^2-1)},$$

$$\sin x = \left\{\frac{2ab+1 \pm \sqrt{(1+4ab-4b^2)}}{2(a^2+1)}\right\}^{\frac{1}{2}}.$$

Ex. 10. $25\sin A(\sin A - \cos A) = 4$. Prove that $\sin A = \frac{4}{5}$.

$$\sin^2 A - \sin A\cos A = \frac{4}{25},$$

$$\sin^2 A - \frac{4}{25} = \sin A \cos A$$

$$= \sin A \sqrt{(1 - \sin^2 A)},$$

$$\sin^4 A - \frac{8}{25}\sin^2 A + \frac{16}{625} = \sin^2 A - \sin^4 A,$$

$$2\sin^4 A - \frac{33}{25}\sin^2 A = -\frac{16}{625},$$

$$\sin^4 A - \frac{33}{50}\sin^2 A = -\frac{8}{625},$$

$$\sin^4 A - \frac{33}{50}\sin^2 A + \left(\frac{33}{100}\right)^2 = \frac{1089}{10000} - \frac{8}{625} = \frac{961}{10000},$$

$$\sin^2 A - \frac{33}{100} = \pm\frac{31}{100},$$

$$\sin^2 A = \frac{33}{100} \pm \frac{31}{100} = \frac{64}{100}, \text{ taking the upper sign;}$$

$$\therefore \sin A = \frac{8}{10} = \frac{4}{5}.$$

Ex. 11. $6\tan A + 12\cot A = 5\sqrt{3}\,.\sec A$. Find $\tan A$.

$$6\tan A + \frac{12}{\tan A} = 5\sqrt{3}\,.\sqrt{(1 + \tan^2 A)},$$

$$36\tan^2 A + 144 + \frac{144}{\tan^2 A} = 75 + 75\tan^2 A,$$

$$39\tan^4 A - 69\tan^2 A = 144,$$

$$13\tan^4 A - 23\tan^2 A = 48,$$

$$\tan^4 A - \frac{23}{13}\tan^2 A = \frac{48}{13},$$

$$\tan^4 A - \frac{23}{13}\tan^2 A + \left(\frac{23}{26}\right)^2 = \frac{529}{676} + \frac{2496}{676} = \frac{3025}{676},$$

$$\tan^2 A - \frac{23}{26} = \pm\frac{55}{26},$$

$$\tan^2 A = \frac{23}{26} \pm \frac{55}{26} = \frac{78}{26} = 3;$$

$$\therefore \tan A = \sqrt{3}; \quad \therefore A = 60^\circ \text{ (p. 14)}.$$

Ex. 12. If $m = \tan A + \sin A$, and $n = \tan A - \sin A$, find an equation involving only m and n.

$$m = \tan A + \sin A,$$
$$n = \tan A - \sin A,$$
$$\tan A = \frac{m+n}{2}, \quad \sin A = \frac{m-n}{2},$$
$$\sin A = \frac{\tan A}{\sqrt{(1+\tan^2 A)}};$$
$$\frac{m-n}{2} = \frac{\frac{m+n}{2}}{\sqrt{\left\{1+\left(\frac{m+n}{2}\right)^2\right\}}};$$
$$\frac{(m-n)^2}{4} = \frac{\frac{(m+n)^2}{4}}{1+\frac{(m+n)^2}{4}} = \frac{(m+n)^2}{4+(m+n)^2},$$
$$4(m-n)^2 + (m-n)^2(m+n)^2 = 4(m+n)^2$$
$$(m^2-n^2)^2 = 4(m+n)^2 - 4(m-n)^2 = 16mn,$$
$$m^2 - n^2 = \sqrt{(16mn)} = 4\sqrt{(mn)}.$$

Ex. 13. If $a(\sin\theta)^2 + b(\cos\theta)^2 = m$ ..... (1),

$b(\sin\phi)^2 + a(\cos\phi)^2 = n$ ..... (2),

and $a \tan\theta = b \tan\phi$ ..... (3),

then $\frac{1}{a} + \frac{1}{b} = \frac{1}{m} + \frac{1}{n}$.

$$a(1-\cos^2\theta) + b\cos^2\theta = m,$$
$$a - a\cos^2\theta + b\cos^2\theta = m,$$
$$(a-b)\cos^2\theta = a - m,$$
$$\cos^2\theta = \frac{a-m}{a-b}; \quad \therefore \sec^2\theta = \frac{a-b}{a-m} \text{ or } 1+\tan^2\theta = \frac{a-b}{a-m};$$
$$\therefore \tan^2\theta = \frac{a-b}{a-m} - 1 = \frac{m-b}{a-m}.$$

In the same manner,

$$\tan^2\phi = \frac{n-a}{b-n};$$

$$\therefore \frac{\tan^2\theta}{\tan^2\phi} = \frac{m-b}{a-m}\cdot\frac{b-n}{n-a},$$

but by equation 3,

$$\frac{\tan\theta}{\tan\phi} = \frac{b}{a}; \quad \frac{\tan^2\theta}{\tan^2\phi} = \frac{b^2}{a^2};$$

$$\therefore \frac{b^2}{a^2} = \frac{m-b}{a-m}\cdot\frac{b-n}{n-a} = \frac{bm-b^2-mn+bn}{an-mn-a^2+am},$$

$$ab^2n-b^2mn-a^2b^2+ab^2m = a^2bm-a^2b^2-a^2mn+a^2bn;$$

$$\therefore (a^2-b^2)\,mn = a^2bm-ab^2m+a^2bn-ab^2n$$
$$= ab\,(am-bm+an-bn)$$
$$= ab\,\{(a-b)\,m+(a-b)\,n\};$$

$$\therefore (a+b)\,mn = ab\,(m+n),$$

$$\therefore \frac{a+b}{ab} = \frac{m+n}{mn};$$

$$\therefore \frac{1}{a}+\frac{1}{b} = \frac{1}{m}+\frac{1}{n}.$$

(1) Prove that

$$\sin A = \frac{\tan A}{\sec A} = \frac{\cos A}{\cot A} = \frac{\tan A\,\cot A}{\operatorname{cosec} A},$$

$$\cos A = \frac{\sin A}{\tan A} = \frac{\cot A}{\operatorname{cosec} A} = \frac{\sin A\,\operatorname{cosec} A}{\sec A},$$

$$\tan A = \frac{\cos A}{\sin A\,\cot^2 A} = \frac{\cos A\,\sec A}{\cot A} = \frac{\sin A\,\operatorname{cosec} A}{\cot A},$$

$$\cot A = \frac{\cos A\,\sec A}{\tan A} = \frac{\sin A\,\operatorname{cosec} A}{\tan A} = \frac{\sin A}{\cos A\,\tan^2 A},$$

$$\sec A = \frac{\tan A}{\sin A} = \frac{\sin A\,\operatorname{cosec} A}{\cos A} = \frac{\tan A\,\cot A}{\cos A},$$

$$\operatorname{cosec} A = \frac{\sec A}{\tan A} = \frac{\cos A\,\sec A}{\sin A} = \frac{\tan A\,\cot A}{\sin A}.$$

(2) $\cos^2 A\,\cot^2 A = \cot^2 A - \cos^2 A.$

(3) $\dfrac{\cos^2 A-\cos^2 B}{\cos^2 A\,\cos^2 B} = \tan^2 B-\tan^2 A.$

(4) If $m = \operatorname{cosec} A - \sin A$; and $n = \sec A - \cos A$;

then $m^{\frac{4}{3}}n^{\frac{2}{3}} + m^{\frac{2}{3}}n^{\frac{4}{3}} = 1.$

(5) Given $\frac{\sin A}{\cos A} = 1$, find A. $A = 45°$.

(6) $\sin A \cos A = \frac{\sqrt{3}}{4}$ find A. $A = 30°$.

(7) $\tan A + \cot A = 2$, find A. $A = 45°$.

(8) Write down the sum of all the exterior and also of all the interior angles of a polygon of n sides; and thence deduce the value of each of the vertical angles of the triangles made by producing both ways all the sides of a regular polygon, and verify the result in the cases where $n = 3$ and $n = 4$.

(9) $\frac{\tan A}{\sqrt{\sec A}} + \sqrt{\left(1 - \frac{1}{\sec A}\right)} = \sec A$, find $\sec A$:

$$\sec A = \frac{1 + \sqrt{5}}{2}.$$

(10) $\sqrt{(\tan^4 A - 1)} + \sqrt[4]{\left(1 - \frac{1}{\tan^4 A}\right)} + 1 = \sec^2 A$, find $\tan A$;

$$\tan A = \left(\frac{1 + \sqrt{2}}{2}\right)^{\frac{1}{4}}.$$

(11) $\sin^2 A + \frac{1}{2}\cos^2 A = 1$, find A.

$$A = 90°.$$

(12) $\sin^4 A - 2\sin^2 A - 1 = 2\sin A - \cos^2 A$, find A.

$$A = 270°.$$

(13) $\tan^4 A - \tan^2 A + 1 = 4\tan A + \sec^2 A$, find $\tan A$.

(14) If $\sin A + \cos B = a$,
$\sin B + \cos A = b$,
then $\sin B = \frac{b}{2} \pm \frac{a}{2}\sqrt{\left(\frac{4}{a^2 + b^2} - 1\right)}$.

(15) Given $\tan x = \frac{3}{\tan x} + \frac{4}{\sqrt{\tan x}} + 3$, find $\tan x$:

$$\tan x = \frac{7 \pm \sqrt{13}}{2}.$$

(16) Given $\left.\begin{aligned}\tan x \tan y &= \tan^2 x - \tan^2 y \\ 2\tan^2 x &= \frac{\tan^2 x + \tan^2 y}{\tan x - \tan y}\end{aligned}\right\}$.

Find $\tan x$ and $\tan y$, without quadratics,

$$\tan x = \frac{\sqrt{5}+5}{4}; \quad \tan y = \frac{\sqrt{5}}{2}.$$

(17) $\left.\begin{aligned}\tan x - \tan y &= 2 \\ \frac{1}{12}(\tan x + \tan y) &= \frac{1}{\tan x \tan y}\end{aligned}\right\}$ $\left.\begin{aligned}\tan x &= 3, \\ \tan y &= 1.\end{aligned}\right.$

(18) $\frac{1}{4}\tan^2 x = \frac{1}{\tan x - 1}$; find $\tan x$:

$$\tan x = 2.$$

(19) $\tan^2 x + 48\cot^2 x = 4\cot x + 18$,

$$\tan x = 4 \text{ and } -2.$$

(20) $\frac{4\sec^2 x}{3} = 65\cot x - 39$; find $\tan x$:

$$\tan x = \frac{3}{2}.$$

(21) $\left.\begin{aligned}\sin^2 x + \sin y &= \frac{3}{2} \\ \sin x \sin y &= \frac{1}{2}\end{aligned}\right\}$ find x and y:

$$x = 90^\circ \text{ and } y = 30^\circ.$$

(22) $\frac{1}{7}\tan^2 x = 3\frac{1}{7} - \tan x$; find $\tan x$:

$$\tan x = 2.$$

(23) If $\tan^2 A - \cot A = \frac{1}{8}(\cot A - 1)$, $A = 45^\circ$.

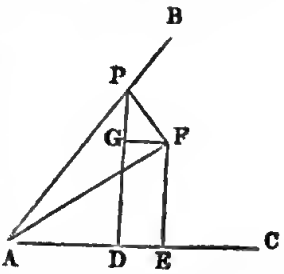
(24) $(\tan A - 1)^2 - 8\cot A(1 - 2\cot A) = 8$: find $\tan A$.

(25) Prove that

$$\frac{1}{\tan A + \dfrac{1}{\tan A + \text{\&c., } ad\ inf.}} = \frac{1}{2}\{\sqrt{(\sec^2 A + 3)} - \tan A\}.$$

CHAPTER II.

7. To find the sine and cosine of the sum and difference of two angles.



Let the angle $FAC = A$, and the angle $FAP = B$, then $PAC = A + B$, draw PF perpendicular to AF, and from P and F let fall the perpendiculars PD and FE on AC, and draw FG parallel to AC. Now, because GF is parallel to AC, the alternate angles, AFG and FAC are equal, but $FAC = A$, therefore $AFG = A$; and since AFP is a right angle, the angle PFG is the complement of AFG; but FGP is a right angle, therefore FPG is the complement of PFG, that is, the angles FPG and AFG are the complements of the same angle (PFG), they are therefore equal; but AFG has been proved equal to A, hence $FPG = A$.

$$\sin(A+B) = \frac{DP}{AP} = \frac{DG+GP}{AP} = \frac{EF+PG}{AP}$$

$$= \frac{EF}{AP} + \frac{PG}{AP}$$

$$= \frac{EF}{AF}\cdot\frac{AF}{AP} + \frac{PG}{PF}\cdot\frac{PF}{AP};$$

but by the definitions, $\frac{EF}{AF} = \sin A$,

and $\frac{AF}{AP} = \cos B$; also $\frac{PG}{PF} = \cos FPG = \cos A$,

and $\frac{PF}{AP} = \sin B$, hence

$$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B.$$

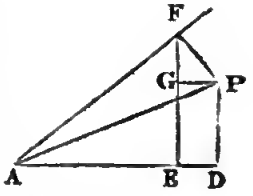
$$\cos(A+B) = \frac{AD}{AP} = \frac{AE-DE}{AP} = \frac{AE}{AP} - \frac{GF}{AP},$$

(because $GF = DE$)

$$= \frac{AE}{AF}\cdot\frac{AF}{AP} - \frac{GF}{PF}\cdot\frac{PF}{AP}$$

$$= \cos A \cos B - \sin A \sin B.$$

To find the sine and cosine of $A - B$.



Let the angle $DAF = A$, and the angle $PAF = B$, then $\angle PAD = A - B$: from any point P draw the perpendiculars PF and PD on AF and AD, also draw PG parallel to AD, then $\angle PFG =$ the complement of AFE, and therefore $= A$, because AFG is the complement of A; and because GD is a rectangle, $ED = PG$, and $PD = GE$.

$$\sin(A-B) = \frac{PD}{AP} = \frac{FE-FG}{AP} = \frac{FE}{AP} - \frac{FG}{AP}$$

$$= \frac{FE}{AF}\cdot\frac{AF}{AP} - \frac{FG}{FP}\cdot\frac{FP}{AP}$$

$$= \sin A \cos B - \cos A \sin B,$$

$$\cos(A-B) = \frac{AD}{AP} = \frac{AE+ED}{AP} = \frac{AE}{AP} + \frac{PG}{AP}$$

$$= \frac{AE}{AF}\cdot\frac{AF}{AP} + \frac{PG}{PF}\cdot\frac{PF}{AP}$$

$$= \cos A \cos B + \sin A \sin B.$$

8. Now, since $\sin(A+B) = \sin A \cos B + \cos A \sin B$, if we make $B = A$, then

$\sin(A+A) = \sin 2A = \sin A \cos A + \cos A \sin A = 2 \sin A \cos A$;

also,

since $\cos(A+B) = \cos A \cos B - \sin A \sin B$; make $B = A$;

then $\cos(A+A) = \cos 2A = \cos A \cos A - \sin A \sin A$

$$= \cos^2 A - \sin^2 A;$$

but since $\cos^2 A = 1 - \sin^2 A$, we have also

$$\cos 2A = 1 - \sin^2 A - \sin^2 A = 1 - 2\sin^2 A;$$

also $\sin^2 A = 1 - \cos^2 A$;

$$\therefore \cos 2A = \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1.$$

Since A may be any angle, we have the sine of any angle equal to twice the sine of half that angle multipled by the cosine of half that angle; and the cosine of any angle is equal to the square of the cosine of half that angle minus the sine square of half that angle. For the sines we have

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2},$$

$$\sin \frac{1}{2} A = 2 \sin \frac{1}{4} A \cos \frac{1}{4} A,$$

$$\sin \frac{1}{3} A = 2 \sin \frac{1}{6} A \cos \frac{1}{6} A,$$

.....

$$\sin \frac{1}{n} A = 2 \sin \frac{1}{2n} A \cos \frac{1}{2n} A.$$

In the same manner for the cosines,

$$\cos 2A = 1 - 2 \sin^2 A,$$

$$\cos A = 1 - 2 \sin^2 \frac{1}{2} A,$$

$$\cos \frac{1}{2} A = 1 - 2 \sin^2 \frac{1}{4} A,$$

$$\cos \frac{1}{3} A = 1 - 2 \sin^2 \frac{1}{6} A,$$

.....

$$\cos \frac{1}{n} A = 1 - 2 \sin^2 \frac{1}{2n} A.$$

Also $\cos 2A = 2 \cos^2 A - 1,$

$$\cos A = 2 \cos^2 \frac{1}{2} A - 1,$$

$$\cos \frac{1}{2} A = 2 \cos^2 \frac{1}{4} A - 1,$$

$$\cos \frac{1}{3} A = 2 \cos^2 \frac{1}{6} A - 1,$$

.....

$$\cos \frac{1}{n} A = 2 \cos^2 \frac{1}{2n} A - 1.$$

9. To find $\sin 3A$.

$$\sin 3A = \sin (2A + A).$$

By putting $2A$ for A, and A for B in the $\sin (A + B)$, we have

$$\sin (2A + A) = \sin 2A \cos A + \cos 2A \sin A;$$

but $\sin 2A = 2 \sin A \cos A$, and $\cos 2A = 1 - 2\sin^2 A$

$$\begin{aligned}\sin(2A + A) &= 2 \sin A \cos A \cos A + (1 - 2\sin^2 A) \sin A,\\ &= 2 \sin A \cos^2 A + \sin A - 2 \sin^3 A,\\ &= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A,\\ &= 2 \sin A - 2 \sin^3 A + \sin A - 2 \sin^3 A,\\ &= 3 \sin A - 4 \sin^3 A;\end{aligned}$$

that is, $\sin 3A = 3 \sin A - 4 \sin^3 A$.

To find $\cos 3A$.

$$\begin{aligned}\cos 3A &= \cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A,\\ &= (2\cos^2 A - 1) \cos A - 2 \sin A \cos A \sin A,\\ &= 2\cos^3 A - \cos A - 2 \cos A \sin^2 A,\\ &= 2\cos^3 A - \cos A - 2 \cos A (1 - \cos^2 A),\\ &= 2\cos^3 A - \cos A - 2 \cos A + 2\cos^3 A,\\ &= 4\cos^3 A - 3 \cos A.\end{aligned}$$

In the same way we may find the $\sin 4A$, &c., but it will be better to express them in general terms, as in the next Article.

10. Now, since

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B,$$

by addition we have

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B \ldots\ldots\ldots(1).$$

By subtraction,

$$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B \ldots\ldots\ldots(2).$$

By multiplication,

$$\begin{aligned}\sin(A + B) \sin(A - B) &= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B\\ &= \sin^2 A (1 - \sin^2 B) - \sin^2 B (1 - \sin^2 A)\\ &= \sin^2 A - \sin^2 B = (\sin A + \sin B)(\sin A - \sin B) \ldots(3).\end{aligned}$$

Also, $\cos(A + B) = \cos A \cos B - \sin A \sin B,$
$\cos(A - B) = \cos A \cos B + \sin A \sin B.$

By addition,

$$\cos(A + B) + \cos(A - B) = 2 \cos A \cos B \ldots\ldots\ldots(4).$$

By subtraction,

$$\cos(A - B) - \cos(A + B) = 2 \sin A \sin B \ldots\ldots\ldots(5).$$

By multiplication,

$$\cos(A+B)\cos(A-B) = \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$$
$$= \cos^2 A(1-\sin^2 B) - \sin^2 B(1-\cos^2 A)$$
$$= \cos^2 A - \sin^2 B$$
$$= (\cos A + \sin B)(\cos A - \sin B) \quad \ldots\ldots\ldots\ldots (6).$$

Now by equations (1) and (4):

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B,$$
$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B;$$

for B put a, and for A put na; then,

$$A+B = na+a = (n+1)a, \text{ and } A-B = na-a = (n-1)a,$$
$$\sin(n+1)a + \sin(n-1)a = 2\sin na\cos a,$$
$$\cos(n+1)a + \cos(n-1)a = 2\cos na\cos a.$$

By giving to n different values, the sine or cosine of any multiple arc may be found.

11. Now, since half the sum of any two quantities added to half their difference gives the greater, and half the difference subtracted from half the sum gives the less, we have

$$A = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B),$$
$$B = \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B).$$

Putting $\frac{1}{2}(A+B)$ for A, and $\frac{1}{2}(A-B)$ for B, in $\sin(A+B)$ and $\cos(A+B)$;

$$\sin A = \sin\{\tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B)\}$$
$$= \sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) + \cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \ldots (1).$$

In the same manner,

$$\sin B = \sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) - \cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \ldots (2).$$

By adding (1) and (2),

$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \quad \ldots\ldots\ldots\ldots (3).$$

By subtracting (2) from (1),

$$\sin A - \sin B = 2\sin\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A+B) \quad \ldots\ldots\ldots\ldots (4).$$

$$\cos A = \cos\{\tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B)\}$$
$$= \cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) - \sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \ldots (5).$$

$$\cos B = \cos\{\tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B)\}$$
$$= \cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) + \sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \ldots (6).$$

Add (5) and (6):

$$\cos A + \cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \quad \ldots\ldots\ldots\ldots(7).$$

Subtract (5) from (6):

$$\cos B - \cos A = 2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \quad \ldots\ldots\ldots\ldots(8).$$

By dividing (3) by (7):

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)} = \frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A+B)}$$
$$= \tan\tfrac{1}{2}(A+B) \ldots\ldots\ldots(9).$$

Divide (4) by (7):

$$\frac{\sin A - \sin B}{\cos A + \cos B} = \frac{2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)}{2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)} = \tan\tfrac{1}{2}(A-B). \quad (10).$$

Divide (3) by (4):

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)}$$
$$= \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} \quad \ldots\ldots\ldots(11).$$

Divide (7) by (8):

$$\frac{\cos B + \cos A}{\cos B - \cos A} = \frac{2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}$$
$$= \cot\tfrac{1}{2}(A+B)\cot\tfrac{1}{2}(A-B) \quad \ldots\ldots\ldots(12).$$

Divide (4) by (8):

$$\frac{\sin A - \sin B}{\cos B - \cos A} = \frac{2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)}{2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}$$
$$= \cot\tfrac{1}{2}(A+B) \quad \ldots\ldots\ldots\ldots\ldots\ldots(13).$$

12. To express the $\tan(A \pm B)$ in terms of $\tan A$ and $\tan B$.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}.$$

Divide the numerator and denominator of this fraction by $\cos A\cos B$; then

$$\tan(A+B) = \frac{\frac{\sin A\cos B}{\cos A\cos B} + \frac{\cos A\sin B}{\cos A\cos B}}{1 - \frac{\sin A\sin B}{\cos A\cos B}} = \frac{\tan A + \tan B}{1 - \tan A\tan B},$$

$$\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}$$
$$=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}-\dfrac{\cos A\sin B}{\cos A\cos B}}{1+\dfrac{\sin A\sin B}{\cos A\cos B}}=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

13. To express the $\cot(A\pm B)$ in terms of $\cot A$ and $\cot B$.

$$\cot(A+B)=\frac{\cos(A+B)}{\sin(A+B)}=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}.$$

Dividing numerator and denominator by $\sin A\sin B$,

$$\cot(A+B)=\frac{\dfrac{\cos A\cos B}{\sin A\sin B}-1}{\dfrac{\sin A\cos B}{\sin A\sin B}+\dfrac{\cos A\sin B}{\sin A\sin B}}=\frac{\cot A\cot B-1}{\cot B+\cot A};$$

making $B=A$, we have

$$\cot 2A=\frac{\cot^2 A-1}{2\cot A}.$$

In precisely the same manner we obtain

$$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}.$$

If in $\tan(A+B)$ we make $B=A$,

$$\text{then } \tan(A+A)=\frac{\tan A+\tan A}{1-\tan A\tan A};$$

$$\text{that is, } \tan 2A=\frac{2\tan A}{1-\tan^2 A}.$$

If for $2A$ we put A, and therefore $\frac{A}{2}$ for A, we have

$$\tan A=\frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}.$$

In the same manner

$$\tan\frac{A}{2}=\frac{2\tan\frac{A}{4}}{1-\tan^2\frac{A}{4}},$$

$$\tan\frac{A}{3}=\frac{2\tan\frac{A}{6}}{1-\tan^2\frac{A}{6}},$$

.....

$$\tan\frac{A}{n}=\frac{2\tan\frac{A}{2n}}{1-\tan^2\frac{A}{2n}}.$$

14. Inverse trigonometrical functions.

The quantities $\sin^{-1}A$, $\cos^{-1}A$, &c., are called inverse functions, and have the following signification:—

$\sin^{-1}A$ is put for an angle whose sine is A,

$\cos^{-1}A$ is put for an angle whose cosine is A, &c., &c.

Let $\sin\alpha=a$, and $\sin\beta=b$,

then $\alpha=\sin^{-1}a$, and $\beta=\sin^{-1}b$,

$\cos\alpha=\sqrt{(1-a^2)}$, and $\cos\beta=\sqrt{(1-b^2)}$,

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$=a\sqrt{(1-b^2)}+b\sqrt{(1-a^2)};$$

$$\therefore\ \alpha+\beta=\sin^{-1}\{a\sqrt{(1-b^2)}+b\sqrt{(1-a^2)}\},$$

but $\alpha=\sin^{-1}a$, and $\beta=\sin^{-1}b$;

$$\therefore\ \sin^{-1}a+\sin^{-1}b=\sin^{-1}\{a\sqrt{(1-b^2)}+b\sqrt{(1-a^2)}\}.$$

Also in the same manner,

$$\sin^{-1}a-\sin^{-1}b=\sin^{-1}\{a\sqrt{(1-b^2)}-b\sqrt{(1-a^2)}\}.$$

By proceeding in the same manner for the cosine,

$$\cos^{-1}a+\cos^{-1}b=\cos^{-1}\{ab-\sqrt{(1-a^2)}\sqrt{(1-b^2)}\},$$
$$\cos^{-1}a-\cos^{-1}b=\cos^{-1}\{ab+\sqrt{(1-a^2)}\sqrt{(1-b^2)}\}.$$

Proceeding in the same way for the tangents,

if $\tan\alpha=a$, $\tan\beta=b$,

then $\alpha=\tan^{-1}a$, and $\beta=\tan^{-1}b$,

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{a+b}{1-ab};$$

$$\therefore\ \alpha+\beta=\tan^{-1}\left(\frac{a+b}{1-ab}\right),$$

but $\alpha=\tan^{-1}a$, and $\beta=\tan^{-1}b$,

$$\therefore \tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\frac{a+b}{1-ab}\right),$$

and since $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \dfrac{a-b}{1+ab}$;

$$\therefore \tan^{-1} a - \tan^{-1} b = \tan^{-1}\left(\frac{a-b}{1+ab}\right).$$

In the same way,

$$\cot^{-1} a \pm \cot^{-1} b = \cot^{-1}\left(\frac{ab \mp 1}{b \pm a}\right).$$

15. The following formulæ, which are sometimes called the formulæ of verification, are very useful:—

since $\cos^2 A + \sin^2 A = 1$,

and $2 \sin A \cos A = \sin 2A$.

By adding and subtracting we have

$$\cos^2 A + 2 \sin A \cos A + \sin^2 A = 1 + \sin 2A,$$

and $$\cos^2 A - 2 \sin A \cos A + \sin^2 A = 1 - \sin 2A.$$

By extracting the square root of each of these,

$$\cos A + \sin A = \sqrt{(1 + \sin 2A)},$$

$$\cos A - \sin A = \sqrt{(1 - \sin 2A)}.$$

When A is less than 45^0, $\sin A$ is less than $\cos A$. By adding,

$$2 \cos A = \sqrt{(1 + \sin 2A)} + \sqrt{(1 - \sin 2A)},$$

$$\cos A = \frac{1}{2}\sqrt{(1 + \sin 2A)} + \frac{1}{2}\sqrt{(1 - \sin 2A)}.$$

Subtracting,

$$2 \sin A = \sqrt{(1 + \sin 2A)} - \sqrt{(1 - \sin 2A)},$$

$$\sin A = \frac{1}{2}\sqrt{(1 + \sin 2A)} - \frac{1}{2}\sqrt{(1 - \sin 2A)}.$$

When A is greater than 45^0, then $\sin A$ is greater than $\cos A$, and the above will become

$$\cos A = \frac{1}{2}\{\sqrt{(1 + \sin 2A)} - \sqrt{(1 - \sin 2A)}\},$$

$$\sin A = \frac{1}{2}\{\sqrt{(1 + \sin 2A)} + \sqrt{(1 - \sin 2A)}\}.$$

To find the $\sin 18^0$.

Since the cosine of any angle is equal to the sine of its complement,

$$\cos 54^0 = \sin 36^0,$$

but $\cos 54^0 = \cos(36^0 + 18^0) = \cos 36^0 \cos 18^0 - \sin 36^0 \sin 18^0$;

$$\therefore \cos 36^0 \cos 18^0 - \sin 36^0 \sin 18^0 = \sin 36^0,$$

but $\cos 36^0 = 1 - 2\sin^2 18^0$, and $\sin 36^0 = 2 \sin 18^0 \cos 18^0$;

$$\therefore \cos 18^0 (1 - 2\sin^2 18^0) - 2\sin^2 18^0 \cos 18^0 = 2 \sin 18^0 \cos 18^0.$$

Dividing by $\cos 18^0$,

$$1 - 2\sin^2 18^0 - 2\sin^2 18^0 = 2 \sin 18^0;$$

$$\therefore 4\sin^2 18^0 + 2 \sin 18^0 = 1,$$

$$\sin^2 18^0 + \frac{1}{2} \sin 18^0 = \frac{1}{4},$$

$$\sin^2 18^0 + \frac{1}{2} \sin 18^0 + \left(\frac{1}{4}\right)^2 = \frac{1}{4} + \frac{1}{16} = \frac{5}{16},$$

$$\sin 18^0 + \frac{1}{4} = \frac{\sqrt{5}}{4},$$

$$\sin 18^0 = \frac{\sqrt{5} - 1}{4}.$$

The same result may be obtained by using the formula for the $\sin 3A$; or by the 10th proposition of the 4th book of Euclid.

$$\cos 18^0 = \sqrt{(1 - \sin^2 18^0)}$$

$$= \sqrt{\left\{1 - \left(\frac{\sqrt{5} - 1}{4}\right)^2\right\}} = \sqrt{\frac{16 - 5 + 2\sqrt{5} - 1}{16}}$$

$$= \sqrt{\frac{5 + \sqrt{5}}{8}} = \frac{\sqrt{(5 + \sqrt{5})}}{2\sqrt{2}},$$

$$\sin 36^0 = 2 \sin 18^0 \cos 18^0$$

$$= \frac{2(\sqrt{5} - 1)}{4} \cdot \frac{\sqrt{(5 + \sqrt{5})}}{2\sqrt{2}} = \frac{(\sqrt{5} - 1)\sqrt{(5 + \sqrt{5})}}{4\sqrt{2}}$$

$$= \frac{\sqrt{\{(5 - 2\sqrt{5} + 1)(5 + \sqrt{5})\}}}{4\sqrt{2}} = \frac{\sqrt{(20 - 4\sqrt{5})}}{4\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} \cdot \sqrt{\frac{20 - 4\sqrt{5}}{4}} = \frac{1}{2\sqrt{2}} \cdot \sqrt{(5 - \sqrt{5})}.$$

Now the $\sin 36^0 = \cos 54^0$,

$$\therefore \cos 54^0 = \frac{1}{2\sqrt{2}} \cdot \sqrt{(5-\sqrt{5})},$$

$$\sin 54^0 = \sqrt{(1-\cos^2 54^0)} = \sqrt{\left(1-\frac{5-\sqrt{5}}{8}\right)}$$

$$= \sqrt{\frac{8-5+\sqrt{5}}{8}} = \sqrt{\frac{3+\sqrt{5}}{8}} = \sqrt{\frac{6+2\sqrt{5}}{16}}$$

$$= \frac{\sqrt{(5+2\sqrt{5}+1)}}{4} = \frac{\sqrt{5}+1}{4}.$$

Or thus, $\sin 54^0 = \cos 36^0 = \cos 2(18^0) = \cos^2 18^0 - \sin^2 18^0$

$$= \frac{5+\sqrt{5}}{8} - \frac{6-2\sqrt{5}}{16} = \frac{10+2\sqrt{5}-6+2\sqrt{5}}{16}$$

$$= \frac{4+4\sqrt{5}}{16} = \frac{1+\sqrt{5}}{4}.$$

Now since $\sin 18^0 = \cos 72^0$, and $\cos 18^0 = \sin 72^0$, we have the following results:

$$\sin 18^0 = \frac{\sqrt{5}-1}{4}; \qquad \cos 18^0 = \frac{\sqrt{(5+\sqrt{5})}}{2\sqrt{2}};$$

$$\sin 36^0 = \frac{\sqrt{(5-\sqrt{5})}}{2\sqrt{2}}; \quad \cos 36^0 = \frac{\sqrt{5}+1}{4};$$

$$\sin 54^0 = \frac{\sqrt{5}+1}{4}; \qquad \cos 54^0 = \frac{\sqrt{(5-\sqrt{5})}}{2\sqrt{2}};$$

$$\sin 72^0 = \frac{\sqrt{(5+\sqrt{5})}}{2\sqrt{2}}; \quad \cos 72^0 = \frac{\sqrt{5}-1}{4}.$$

By the formula, page 34, we may find the $\sin 15^0$.

$$\sin 15^0 = \frac{1}{2}\{\sqrt{(1+\sin 30^0)} - \sqrt{(1-\sin 30^0)}\}$$

$$= \frac{1}{2}\left\{\sqrt{\left(1+\frac{1}{2}\right)} - \left\{\sqrt{\left(1-\frac{1}{2}\right)}\right\}\right.$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

This might have been found by putting

$$\sin 15^0 = \sin (45^0 - 30^0).$$

Examples.

(1) Shew that

$$\cos(60^\circ + A) + \cos(60^\circ - A) = \cos A.$$

$$\cos(60^\circ + A) = \cos 60^\circ \cos A - \sin 60^\circ \sin A,$$

$$\cos(60^\circ - A) = \cos 60^\circ \cos A + \sin 60^\circ \sin A.$$

By addition,

$$\cos(60^\circ + A) + \cos(60^\circ - A) = 2\cos 60^\circ \cos A = 2 \times \frac{1}{2} \cos A$$

$$= \cos A \left(\text{because } \cos 60^\circ = \frac{1}{2}\right).$$

(2) $\sin(30^\circ + A) + \sin(30^\circ - A) = \cos A.$

$$\sin(30^\circ + A) = \sin 30^\circ \cos A + \cos 30^\circ \sin A,$$

$$\sin(30^\circ - A) = \sin 30^\circ \cos A - \cos 30^\circ \sin A.$$

By addition,

$$\sin(30^\circ + A) + \sin(30^\circ - A) = 2\sin 30^\circ \cos A = \cos A.$$

(3) $\sin A = \dfrac{2\tan \frac{1}{2}A}{1 + \tan^2 \frac{1}{2}A}$;

$$\sin A = \frac{\sin A}{1} = \frac{2\sin \frac{1}{2}A \cos \frac{1}{2}A}{1} = \frac{\dfrac{2\sin \frac{1}{2}A \cos \frac{1}{2}A}{\cos^2 \frac{1}{2}A}}{\dfrac{1}{\cos^2 \frac{1}{2}A}},$$

(by dividing numerator and denominator by $\cos^2 \frac{1}{2}A$)

$$= \frac{2\tan \frac{1}{2}A}{\sec^2 \frac{1}{2}A} = \frac{2\tan \frac{1}{2}A}{1 + \tan^2 \frac{1}{2}A}.$$

(4) $2\cos(45^\circ + \frac{1}{2}A)\cos(45^\circ - \frac{1}{2}A) = \cos A.$

$$\cos(45^\circ + \tfrac{1}{2}A) = \cos 45^\circ \cos \tfrac{1}{2}A - \sin 45^\circ \sin \tfrac{1}{2}A$$

$$= \frac{1}{\sqrt{2}}(\cos \tfrac{1}{2}A - \sin \tfrac{1}{2}A);$$

$$\cos(45^\circ - \tfrac{1}{2}A) = \cos 45^\circ \cos \tfrac{1}{2}A + \sin 45^\circ \sin \tfrac{1}{2}A$$

$$= \frac{1}{\sqrt{2}}(\cos \tfrac{1}{2}A + \sin \tfrac{1}{2}A).$$

By multiplication,

$$2\cos(45^\circ + \tfrac{1}{2}A)\cos(45^\circ - \tfrac{1}{2}A)$$

$$= 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} (\cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A)$$

$$= \cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A = \cos A.$$

(5) Given

$\sin(x+a)+\cos(x+a)=\sin(x-a)+\cos(x-a)$, find x.

$\sin(x+a)-\sin(x-a)=\cos(x-a)-\cos(x+a)$,

$\sin(x+a)=\sin x\cos a+\sin a\cos x$,

$\sin(x-a)=\sin x\cos a-\sin a\cos x$;

$\therefore \sin(x+a)-\sin(x-a)=2\sin a\cos x$,

$\cos(x-a)=\cos x\cos a+\sin x\sin a$,

$\cos(x+a)=\cos x\cos a-\sin x\sin a$;

$\therefore \cos(x-a)-\cos(x+a)=2\sin x\sin a$;

hence, $2\sin a\cos x=2\sin a\sin x$;

$\therefore \cos x=\sin x$;

$\therefore x=45^0$.

(6) If $\sin x+\cos 2x=\sqrt{\frac{5}{4}}$; prove that $x=18^0$.

Since $\cos 2x=1-2\sin^2 x$, we have

$$\sin x+1-2\sin^2 x=\frac{\sqrt{5}}{2},$$

$$\sin^2 x-\frac{1}{2}\sin x=\frac{2-\sqrt{5}}{4};$$

$$\sin^2 x-\frac{1}{2}\sin x+\frac{1}{16}=\frac{2-\sqrt{5}}{4}+\frac{1}{16}=\frac{8-4\sqrt{5}}{16}+\frac{1}{16}$$

$$=\frac{9-4\sqrt{5}}{16}=\frac{(\sqrt{5}-2)^2}{16},$$

$$\sin x-\frac{1}{4}=\frac{\sqrt{5}-2}{4},$$

$$\sin x=\frac{\sqrt{5}-2}{4}+\frac{1}{4}=\frac{\sqrt{5}-1}{4}=\sin 18^0;$$

$\therefore x=18^0$.

(7) Find the tangent of 15^0.

$$\tan 15^0=\tan(45^0-30^0)=\frac{\tan 45^0-\tan 30^0}{1+\tan 45^0\tan 30^0}$$

$$=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{(\sqrt{3}-1)^2}{2}=2-\sqrt{3}.$$

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(8) If $x = \sin^{-1}\frac{3}{5}$, and $y = \sin^{-1}\frac{4}{5}$, then, $x + y = 90^0$.

Since $\sin x = \frac{3}{5}$; $\cos x = \sqrt{\left\{1 - \left(\frac{3}{5}\right)^2\right\}} = \frac{4}{5}$,

and $\sin y = \frac{4}{5}$; $\cos y = \sqrt{\left\{1 - \left(\frac{4}{5}\right)^2\right\}} = \frac{3}{5}$,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

$$= \frac{3}{5}\times\frac{3}{5} + \frac{4}{5}\times\frac{4}{5} = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1,$$

but $\sin 90^0 = 1$; $\therefore\ x + y = 90^0$,

or we may put it in the following form,

$$\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = \sin^{-1}\left(\frac{3}{5}\times\frac{3}{5} + \frac{4}{5}\times\frac{4}{5}\right) = \sin^{-1} 1.$$

(9) Shew that $4 \sin 9^0 = \sqrt{(3 + \sqrt{5})} - \sqrt{(5 - \sqrt{5})}$.

By Art. 21, $\sin A = \frac{1}{2}\{\sqrt{(1 + \sin 2A)} - \sqrt{(1 - \sin 2A)}\}$,

we use the under sign as the angle is less than 45^0.

Now, in the above formula let $A = 9^0$, then, $2A = 18^0$,

$$\sin 9^0 = \frac{1}{2}\{\sqrt{(1 + \sin 18^0)} - \sqrt{(1 - \sin 18^0)}\}$$

$$= \frac{1}{2}\left\{\sqrt{\left(1 + \frac{\sqrt{5}-1}{4}\right)} - \sqrt{\left(1 - \frac{\sqrt{5}-1}{4}\right)}\right\}$$

$$= \frac{1}{4}\{\sqrt{(3 + \sqrt{5})} - \sqrt{(5 - \sqrt{5})}\};$$

$$\therefore\ 4 \sin 9^0 = \sqrt{(3 + \sqrt{5})} - \sqrt{(5 - \sqrt{5})}.$$

(10) Shew that $2 \cos 11^0 15' = \sqrt{\{2 + \sqrt{(2 + \sqrt{2})}\}}$.

Since $2\cos^2 \frac{1}{2}A - 1 = \cos A$,

$$\cos \tfrac{1}{2}A = \sqrt{\frac{1 + \cos A}{2}},$$

let $A = 45^0$, then $\frac{1}{2}A = 22^0 30'$,

$$\cos 22^0 30' = \sqrt{\frac{1 + \cos 45^0}{2}} = \frac{\sqrt{1 + \frac{1}{\sqrt{2}}}}{2}$$

$$= \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} = \sqrt{\frac{2 + \sqrt{2}}{4}} = \frac{\sqrt{(2 + \sqrt{2})}}{2},$$

$$\cos 11^\circ 15' = \sqrt{\frac{1+\cos 22^\circ 30'}{2}} = \frac{\sqrt{1+\frac{\sqrt{(2+\sqrt{2})}}{2}}}{2}$$

$$= \sqrt{\frac{2+\sqrt{(2+\sqrt{2})}}{4}} = \frac{\sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}}{2};$$

$$\therefore\ 2\cos 11^\circ 15' = \sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}.$$

Or it might have been put in this form:

$$2\cos^2\frac{45^\circ}{2} - 1 = \cos 45^\circ = \frac{1}{\sqrt{2}},$$

$$2\left(2\cos^2\frac{45^\circ}{2} - 1\right) = \frac{2}{\sqrt{2}} = \sqrt{2},$$

$$4\cos^2\frac{45^\circ}{2} = 2+\sqrt{2},$$

$$2\cos\frac{45^\circ}{2} = \sqrt{(2+\sqrt{2})},$$

$$2\left(2\cos^2\frac{45^\circ}{4} - 1\right) = \sqrt{(2+\sqrt{2})};$$

$$\therefore\ 2\cos\frac{45^\circ}{4} = \sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}.$$

This is the same as $2\cos\frac{45^\circ}{2^2} = \sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}$, and by continuing the operation to $\frac{45^\circ}{2^n}$, we have

$$2\cos\frac{45^\circ}{2^n} = \sqrt{\{2+\sqrt{(2+\text{to } n+1 \text{ terms})}\}}.$$

(11) If A, B, and C be in arithmetical progression,

$$\sin A - \sin B = 2\sin(A-B)\cos B.$$

Since B is an arithmetical mean between A and C, we have

$$\tfrac{1}{2}(A+C) = B \ldots\ldots (1),\quad \tfrac{1}{2}(A-C) = B-C \ldots\ldots (2),$$

$$A-B = B-C \ldots\ldots\ldots (3).$$

By page 29,

$$\sin A - \sin C = 2\sin\tfrac{1}{2}(A-C)\cos\tfrac{1}{2}(A+C)$$

$$= 2\sin(B-C)\cos B;\ \text{by (2) and (1)}$$

$$= 2\sin(A-B)\cos B;\ \text{by (3)}.$$

(12) Show that

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=45^{0}.$$

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}$$

$$=\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}\right)+\left(\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}\right),$$

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}=\tan^{-1}\left\{\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\cdot\frac{1}{5}}\right\}=\tan^{-1}\frac{4}{7},$$

$$\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\tan^{-1}\left\{\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\cdot\frac{1}{8}}\right\}=\tan^{-1}\frac{3}{11};$$

$$\therefore\ \tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\tan^{-1}\frac{4}{7}+\tan^{-1}\frac{3}{11},$$

$$\text{but } \tan^{-1}\frac{4}{7}+\tan^{-1}\frac{3}{11}=\tan^{-1}\left\{\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\cdot\frac{3}{11}}\right\}=\tan^{-1}(1)=45^{0};$$

$$\therefore\ \tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=45^{0}.$$

(13) Given $\text{vers}^{-1}\frac{x}{a}-\text{vers}^{-1}(1-b)=\text{vers}^{-1}\frac{bx}{a}$; find $\frac{x}{a}$

$$\cos^{-1}\left(1-\frac{x}{a}\right)-\cos^{-1}\{1-(1-b)\}=\cos^{-1}\left(1-\frac{bx}{a}\right),$$

$$\therefore \text{ p. 32, } \left\{1-\frac{x}{a}\right\}b+\sqrt{(1-b^{2})}\sqrt{\left\{1-\left(1-\frac{x}{a}\right)^{2}\right\}}=1-\frac{bx}{a};$$

$$\therefore\ \frac{1-b}{\sqrt{(1-b^{2})}}=\sqrt{\frac{1-b}{1+b}}=\sqrt{\left\{1-\left(1-\frac{x}{a}\right)^{2}\right\}};$$

$$\therefore\ \left(1-\frac{x}{a}\right)^{2}=1-\frac{1-b}{1+b}=\frac{2b}{1+b},$$

$$1-\frac{x}{a}=\pm\sqrt{\frac{2b}{1+b}};$$

$$\therefore \frac{x}{a} = 1 \mp \sqrt{\frac{2b}{1+b}}.$$

(14) If $\sin x = \frac{1}{\sqrt{3}}$ and $\cos y = \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{3}}$, then $x - y = 30°$.

$$\cos y = \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{3}} = \frac{\sqrt{6}+1}{2\sqrt{3}}$$

$$\sin y = \sqrt{\left\{1 - \left(\frac{\sqrt{6}+1}{2\sqrt{3}}\right)^2\right\}} = \frac{\sqrt{(5-2\sqrt{6})}}{2\sqrt{3}}$$

$$\cos x = \sqrt{\left(1 - \frac{1}{3}\right)} = \sqrt{\frac{2}{3}}$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$= \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{6}+1}{2\sqrt{3}} - \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{(5-2\sqrt{6})}}{2\sqrt{3}}$$

$$= \frac{\sqrt{6}+1}{6} - \frac{\sqrt{2}.\sqrt{(5-2\sqrt{6})}}{6}$$

$$= \frac{\sqrt{6}+1-\sqrt{(10-4\sqrt{6})}}{6} = \frac{\sqrt{6}+1-\sqrt{(4-4\sqrt{6}+6)}}{6}$$

$$= \frac{\sqrt{6}+1-\sqrt{(\sqrt{6}-2)^2}}{6} = \frac{\sqrt{6}+1-(\sqrt{6}-2)}{6} = \frac{3}{6} = \frac{1}{2} = \sin 30°.$$

Hence $x - y = 30°$.

(15) If $4 \sin x \sin 3x = 1$, find x.

$$4 \sin x(3 \sin x - 4 \sin^3 x) = 1,$$

$$12 \sin^2 x - 16 \sin^4 x = 1,$$

$$\sin^4 x - \frac{3}{4} \sin^2 x = -\frac{1}{16},$$

$$\sin^4 x - \frac{3}{4} \sin^2 x + \frac{9}{64} = \frac{9}{64} - \frac{1}{16} = \frac{5}{64},$$

$$\sin^2 x - \frac{3}{8} = \pm \frac{\sqrt{5}}{8}$$

$$\sin^2 x = \frac{3 \pm \sqrt{5}}{8} = \frac{6 \pm 2\sqrt{5}}{16} = \frac{(\sqrt{5} \pm 1)^2}{16},$$

$\sin x = \frac{\sqrt{5} \pm 1}{4}$ taking the upper sign, $x = 54°$, and taking the under sign, $x = 18°$.

(16) If $\cos x = \sqrt{\frac{2}{3}}$ and $\cos y = \frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}}$, show that $x+y=60^\circ$.

$$\sin x = \sqrt{\left(1-\frac{2}{3}\right)} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}},$$

$$\sin y = \sqrt{\left\{1-\left(\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}}\right)^2\right\}} = \frac{\sqrt{(7-2\sqrt{6})}}{2\sqrt{3}}$$

$$= \frac{\sqrt{(\sqrt{6}-1)^2}}{2\sqrt{3}} = \frac{\sqrt{6}-1}{2\sqrt{3}},$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$= \sqrt{\frac{2}{3}}\cdot\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}} - \frac{1}{\sqrt{3}}\cdot\frac{\sqrt{6}-1}{2\sqrt{3}}$$

$$\frac{\sqrt{6}+2}{6} - \frac{\sqrt{6}-1}{6} = \frac{\sqrt{6}+2-\sqrt{6}+1}{6} = \frac{3}{2}\cdot\frac{1}{2} = \cos 60^\circ.$$

(17) If $\tan 2x = \frac{2\tan x}{1-4\tan^2 x}$, show that the radius of the arc x is $\frac{1}{2}$.

$$\text{Since } \tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

$$\tan 2x = \frac{2r^2 \tan x}{r^2 - \tan^2 x} \text{ to radius } r;$$

$$\therefore \frac{2r^2\tan x}{r^2-\tan^2 x} = \frac{2\tan x}{1-4\tan^2 x}$$

$$\frac{r^2}{r^2-\tan^2 x} = \frac{1}{1-4\tan^2 x}$$

$$r^2 - 4r^2\tan^2 x = r^2 - \tan^2 x$$

$$4r^2 = 1$$

$$r^2 = \frac{1}{4}; \therefore r = \frac{1}{2}.$$

(18) If $\sin(\alpha+\theta)\sin\theta = \cos^2\frac{\alpha}{2}$, show that $\theta = 90^\circ - \frac{\alpha}{2}$.

$$\text{Since } \cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2},$$

$$(\sin\alpha\cos\theta + \cos\alpha\sin\theta)\sin\theta = \frac{1+\cos\alpha}{2},$$

$$\sin\alpha\sin\theta\cos\theta + \cos\alpha\sin^2\theta = \frac{1+\cos\alpha}{2},$$

$$\frac{1}{2}\sin\alpha\sin 2\theta + \frac{1-\cos 2\theta}{2}\cos\alpha = \frac{1+\cos\alpha}{2},$$

$$\sin\alpha\sin 2\theta + (1-\cos 2\theta)\cos\alpha = 1+\cos\alpha,$$

$$\sin\alpha\sin 2\theta - \cos\alpha\cos 2\theta = 1;$$

$$\text{or, } \cos\alpha\cos 2\theta - \sin\alpha\sin 2\theta = -1;$$

$$i.\,e.,\ \cos(\alpha+2\theta) = -1;$$

$$\therefore\ \alpha + 2\theta = 180^\circ;$$

$$\therefore\ \theta = \frac{180^\circ-\alpha}{2} = 90^\circ - \frac{\alpha}{2}.$$

(19) Cos $n\theta$ + cos $(n-2)\,\theta$ = cos θ, then $\theta = \dfrac{60^\circ}{n-1}$.

$$\cos\{(n-1)+1\}\,\theta + \cos\{(n-1)-1\}\,\theta = \cos\theta,$$

$$\cos(n-1)\,\theta\cos\theta - \sin(n-1)\,\theta\sin\theta + \cos(n-1)\,\theta\cos\theta + \sin(n-1)\,\theta\sin\theta = \cos\theta;$$

$$\therefore\ 2\cos(n-1)\,\theta = 1,$$

$$\cos(n-1)\,\theta = \frac{1}{2},$$

$$(n-1)\,\theta = 60^\circ;$$

$$\therefore\ \theta = \frac{60^\circ}{n-1}.$$

(20) Show that

$$\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3 = 45^\circ.$$

$$\text{Let } z = \cot^{-1}3;\ \ \therefore\ \cot z = 3,$$

$$\frac{\cos^2 z}{\sin^2 z} = 9 \text{ or } \frac{1-\sin^2 z}{\sin^2 z} = 9;\ \ \therefore\ \sin^2 z = \frac{1}{10} \text{ and } \sin z = \frac{1}{\sqrt{10}};$$

$$\text{the above becomes } \sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}} = 45^\circ,$$

$$\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}} = \sin^{-1}\left(\frac{1}{\sqrt{5}}\cdot\frac{3}{\sqrt{10}} + \frac{1}{\sqrt{10}}\cdot\frac{2}{\sqrt{5}}\right)$$

$$= \sin^{-1}\left(\frac{3}{\sqrt{50}} + \frac{2}{\sqrt{50}}\right) = \sin^{-1}\frac{5}{\sqrt{50}} = \sin^{-1}\frac{1}{\sqrt{2}} = 45^\circ.$$

(21) Show that

$$\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3} = 45^\circ.$$

This is the same as $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{3}=45^0$,

$$\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}=\tan^{-1}\left\{\frac{\frac{1}{7}+\frac{1}{3}}{1-\frac{1}{7}\cdot\frac{1}{3}}\right\}=\tan^{-1}\left\{\frac{\frac{10}{21}}{\frac{20}{21}}\right\}=\tan^{-1}\frac{1}{2},$$

$$\text{then } \tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\left\{\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right\}=\tan^{-1}\left\{\frac{\frac{5}{6}}{\frac{5}{6}}\right\}$$

$$=\tan^{-1}1=45^0.$$

(22) Find $\sin A$ from the equation

$$\tan\tfrac{1}{2}A=\operatorname{cosec}A-\sin A.$$

$$\sin A=\operatorname{cosec}A-\tan\tfrac{1}{2}A=\frac{1}{\sin A}-\tan\tfrac{1}{2}A,$$

$$=\frac{1}{2\sin\frac{1}{2}A\cos\frac{1}{2}A}-\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A}=\frac{1}{2\sin\frac{1}{2}A\cos\frac{1}{2}A}-\frac{2\sin^2\frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A}$$

$$=\frac{1-2\sin^2\frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A}=\frac{\cos A}{\sin A};$$

$$\sin^2 A=\cos A=\sqrt{(1-\sin^2 A)},$$

$$\sin^4 A=1-\sin^2 A,$$

$$\sin^4 A+\sin^2 A=1,$$

$$\sin^4 A+\sin^2 A+\frac{1}{4}=1+\frac{1}{4}=\frac{5}{4},$$

$$\sin^2 A+\frac{1}{2}=\frac{\sqrt{5}}{2},$$

$$\sin^2 A=-\frac{1}{2}\pm\frac{\sqrt{5}}{2},$$

$$\sin A=\sqrt{\frac{-1\pm\sqrt{5}}{2}}.$$

(23) $\cot^2(45^0+\frac{1}{2}A)=\dfrac{2\operatorname{cosec}2A-\sec A}{2\operatorname{cosec}2A+\sec A}$.

$$\cot^2(45^0+\tfrac{1}{2}A)=\frac{(\cos\frac{1}{2}A-\sin\frac{1}{2}A)^2}{(\cos\frac{1}{2}A+\sin\frac{1}{2}A)^2}$$

$$=\frac{\cos^2\frac{1}{2}A-2\sin\frac{1}{2}A\cos\frac{1}{2}A+\sin^2\frac{1}{2}A}{\cos^2\frac{1}{2}A+2\sin\frac{1}{2}A\cos\frac{1}{2}A+\sin^2\frac{1}{2}A}$$

$$= \frac{1 - 2\sin\frac{1}{2}A\cos\frac{1}{2}A}{1 + 2\sin\frac{1}{2}A\cos\frac{1}{2}A} = \frac{1 - \sin A}{1 + \sin A}$$

$$= \frac{1 - \dfrac{1}{\operatorname{cosec} A}}{1 + \dfrac{1}{\operatorname{cosec} A}} = \frac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1}$$

$$= \frac{\sec A \operatorname{cosec} A - \sec A}{\sec A \operatorname{cosec} A + \sec A}$$

$$= \frac{2\operatorname{cosec} 2A - \sec A}{2\operatorname{cosec} 2A + \sec A}.$$

(24) Prove $\dfrac{2\sin A - \sin 2A}{2\sin A + \sin 2A} = \tan^2 \frac{1}{2}A.$

$$\frac{2\sin A - \sin 2A}{2\sin A + \sin 2A} = \frac{2\sin A - 2\sin A\cos A}{2\sin A + 2\sin A\cos A} = \frac{1 - \cos A}{1 + \cos A}$$

$$= \frac{1 - (1 - 2\sin^2\frac{1}{2}A)}{1 + \cos^2\frac{1}{2}A - 1} = \frac{2\sin^2\frac{1}{2}A}{2\cos^2\frac{1}{2}A} = \frac{\sin^2\frac{1}{2}A}{\cos^2\frac{1}{2}A} = \tan^2\frac{1}{2}A.$$

(25) Prove that $\dfrac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \tan 3A.$

By formula, page 29,

$$\sin A + \sin 5A = 2\sin 3A\cos 2A,$$

$$\cos A + \cos 5A = 2\cos 3A\cos 2A,$$

$$\sin A + \sin 3A + \sin 5A = \sin 3A + 2\sin 3A\cos 2A,$$

$$= \sin 3A\,(1 + 2\cos 2A),$$

$$\cos A + \cos 3A + \cos 5A = \cos 3A + 2\cos 3A\cos 2A$$

$$= \cos 3A\,(1 + 2\cos 2A);$$

$$\therefore \frac{\sin A + \sin 3A + \sin 5A}{\cos A + \cos 3A + \cos 5A} = \frac{\sin 3A\,(1 + 2\cos 2A)}{\cos 3A\,(1 + 2\cos 2A)}$$

$$= \tan 3A.$$

(26) If $A + B + C = 90^\circ$, prove that

$$\cot A + \cot B + \cot C = \cot A\cot B\cot C.$$

By Art. 19, $\cot(A + B) = \dfrac{\cot A\cot B - 1}{\cot B + \cot A}.$

Put $A+B$ for A, and C for B, and we have

$$\cot(A+B+C)=\frac{\cot(A+B)\cot C-1}{\cot C+\cot(A+B)}=0,$$

because the $\cot 90^0=0$;

$$\therefore \cot(A+B)\cot C-1=0,$$
$$\cot(A+B)\cot C=1,$$
$$\text{or } \frac{\cot A\cot B-1}{\cot B+\cot A}\cot C=1,$$
$$\cot A\cot B\cot C-\cot C=\cot A+\cot B;$$
$$\therefore \cot A\cot B\cot C=\cot A+\cot B+\cot C.$$

(27) If $A+B+C=180^0$, prove that

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1,$$
$$\cos A=-\cos(B+C)=-\cos B\cos C+\sin B\sin C,$$
$$\cos^2 A=-\cos A\cos B\cos C+\sin B\sin C\cos A,$$
$$\cos^2 B=-\cos A\cos B\cos C+\sin A\sin C\cos B,$$
$$\cos^2 A+\cos^2 B$$
$$=-2\cos A\cos B\cos C+\sin C(\sin B\cos A+\sin A\cos B),$$
$$\cos^2 A+\cos^2 B+2\cos A\cos B\cos C=\sin C\sin(A+B)$$
$$=\sin C\sin C=\sin^2 C$$
$$=1-\cos^2 C;$$
$$\therefore \cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1.$$

On the same supposition,

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C,$$
$$\sin 2A+\sin 2B+\sin 2C$$
$$=2\sin A\cos A+2\sin B\cos B+2\sin C\cos C,$$

but $\cos A=-\cos(B+C)$, $\cos C=-\cos(A+B)$,

and $\sin B=\sin(A+C)$;

$$\therefore \sin 2A+\sin 2B+\sin 2C=-2\sin A\cos(B+C)$$
$$+2\cos B\sin(A+C)-2\sin C\cos(A+B)$$

= (by expanding)

$$-2\sin A\cos B\cos C+2\sin A\sin B\sin C,$$
$$+2\sin A\cos B\cos C+2\sin A\sin B\sin C,$$
$$-2\cos A\cos B\sin C,$$
$$+2\cos A\cos B\sin C,$$
$$=4\sin A\sin B\sin C.$$

Also, $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C$:

$$\sin C = \sin (A + B),$$

$$\sin A + \sin B = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B),$$

$$\sin C = \sin (A + B) = 2 \sin \tfrac{1}{2}(A + B) \cos (A + B),$$

$$\tfrac{1}{2}(A + B) = 90^0 - \tfrac{1}{2} C,$$

$$\sin \tfrac{1}{2}(A + B) = \sin (90^0 - \tfrac{1}{2} C) = \cos \tfrac{1}{2} C,$$

$$\sin A + \sin B + \sin C = 2 \cos \tfrac{1}{2} C \{\cos \tfrac{1}{2}(A + B) + \cos \tfrac{1}{2}(A - B)\},$$

but $\cos \frac{1}{2}(A + B) + \cos \frac{1}{2}(A - B) = 2 \cos \frac{1}{2} A \cos \frac{1}{2} B$,

$$\therefore \sin A + \sin B + \sin C = 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C.$$

Or thus; since $A + B + C = 180^0$, then $\frac{1}{2} A + \frac{1}{2} B + \frac{1}{2} C = 90^0$,

$$\begin{aligned}
&\sin A + \sin B + \sin C \\
&\quad = 2 \sin \tfrac{1}{2} A \cos \tfrac{1}{2} A + 2 \sin \tfrac{1}{2} B \cos \tfrac{1}{2} B + 2 \sin \tfrac{1}{2} C \cos \tfrac{1}{2} C \\
&\quad = 2 \cos \tfrac{1}{2} A \cos (\tfrac{1}{2} B + \tfrac{1}{2} C) + 2 \sin \tfrac{1}{2} B \sin (\tfrac{1}{2} A + \tfrac{1}{2} C) \\
&\quad + 2 \cos \tfrac{1}{2} C \cos (\tfrac{1}{2} A + \tfrac{1}{2} B),
\end{aligned}$$

= (by expanding)

$$\begin{aligned}
&2 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C - 2 \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C \cos \tfrac{1}{2} A \\
&+ 2 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C + 2 \sin \tfrac{1}{2} B \sin \tfrac{1}{2} C \cos \tfrac{1}{2} A \\
&\qquad + 2 \sin \tfrac{1}{2} B \sin \tfrac{1}{2} A \cos \tfrac{1}{2} C \\
&\qquad - 2 \sin \tfrac{1}{2} B \sin \tfrac{1}{2} A \cos \tfrac{1}{2} C \\
&= 4 \cos \tfrac{1}{2} A \cos \tfrac{1}{2} B \cos \tfrac{1}{2} C.
\end{aligned}$$

(28) If $\sin 2x = (\sin 3x)^2$, show that $x = 15^0$.

Subtracting each side from unity, $1 - \sin 2x = 1 - \sin^2 3x$,

$$\cos^2 x - 2 \sin x \cos x + \sin^2 x = \cos^2 3x,$$

$$\cos x - \sin x = \cos 3x = 4 \cos^3 x - 3 \cos x,$$

$$\sin x = 4 \cos x (1 - \cos^2 x) = 4 \cos x \sin^2 x;$$

$$\therefore \sin x = 0. \quad \therefore x = 0, \text{ or } \pi,$$

$$\text{and } 2 \sin x \cos x = \frac{1}{2},$$

$$\text{or } \sin 2x = \frac{1}{2} = \sin 30^0;$$

$$\therefore x = 15^0.$$

(29) $\left.\begin{aligned} \cos x + \cos y &= \frac{1}{2}(\sqrt{2} + \sqrt{3}) \\ \cos 3x + \cos 3y &= -\frac{1}{\sqrt{2}} \end{aligned}\right\}$, find x and y.

The second equation is the same as

$$4(\cos^3 x + \cos^3 y) - 3(\cos x + \cos y) = -\frac{1}{\sqrt{2}}; \text{ (page 28.)}$$

$$\therefore \cos^3 x + \cos^3 y = \frac{1}{4}\left\{\frac{3}{\sqrt{2}} + \frac{3\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\right\},$$

$$\text{or } (\cos x + \cos y)(\cos^2 x - \cos x \cos y + \cos^2 y) = \frac{1}{4}\left(\sqrt{2} + \frac{3\sqrt{3}}{2}\right);$$

$$\therefore \cos^2 x - \cos x \cos y + \cos^2 y = \frac{1}{4}\left(\frac{2\sqrt{2} + 3\sqrt{3}}{\sqrt{2} + \sqrt{3}}\right)$$

$$= \frac{2\sqrt{6} + 9 - 4 - 3\sqrt{6}}{4}, \text{ by multiplying by } (\sqrt{3} - \sqrt{2});$$

$$\therefore \cos^2 x - \cos x \cos y + \cos^2 y = \frac{1}{4}(5 - \sqrt{6}),$$

$$\text{but } \cos^2 x + 2\cos x \cos y + \cos^2 y = \frac{1}{4}(5 + 2\sqrt{6});$$

$$\therefore 3\cos x \cos y = \frac{1}{4}.3\sqrt{6};$$

$$\therefore \cos x \cos y = \frac{1}{4}\sqrt{6};$$

$$\therefore \cos^2 x - 2\cos x \cos y + \cos^2 y = \frac{1}{4}(5 - 2\sqrt{6});$$

$$\therefore \cos x - \cos y = \pm\frac{1}{2}(\sqrt{3} - \sqrt{2});$$

$$\text{but } \cos x + \cos y = \frac{1}{2}(\sqrt{3} + \sqrt{2}).$$

By adding these equations, we have

$$\cos x = \frac{\sqrt{3}}{2}, \text{ or } \frac{1}{\sqrt{2}}; \therefore x = 30^\circ, \text{ or } 45^\circ.$$

By subtracting them

$$\cos y = \frac{1}{\sqrt{2}}, \text{ or } \frac{\sqrt{3}}{2}; \therefore y = 45^\circ, \text{ or } 30^\circ.$$

(1) Prove the following theorems,

$$\sin A = \frac{2\cos\frac{1}{2}A}{\operatorname{cosec}\frac{1}{2}A} = \frac{2\tan\frac{1}{2}A}{\sec^2\frac{1}{2}A} = \frac{2}{\tan\frac{1}{2}A + \cot\frac{1}{2}A}$$

$$= \frac{1}{\cot\frac{1}{2}A - \cot A};$$

$$\cos A = \frac{\sin 2A}{2\sin A} = \frac{\operatorname{cosec}\frac{1}{2}A - 2\sin\frac{1}{2}A}{\operatorname{cosec}\frac{1}{2}A} = \frac{\cot\frac{1}{2}A - \tan\frac{1}{2}A}{\cot\frac{1}{2}A + \tan\frac{1}{2}A}$$

$$= \frac{1}{1 + \tan A \tan\frac{1}{2}A};$$

$$\tan A = \frac{2}{\cot\frac{1}{2}A - \tan\frac{1}{2}A} = \frac{2\cot\frac{1}{2}A}{\cot^2\frac{1}{2}A - 1} = \frac{1 + \sec A}{\cot\frac{1}{2}A};$$

$$\sec A = \frac{\sec^2\frac{1}{2}A}{2 - \sec^2\frac{1}{2}A} = \frac{\operatorname{cosec}^2\frac{1}{2}A}{\operatorname{cosec}^2\frac{1}{2}A - 2} = \frac{\cot\frac{1}{2}A + \tan\frac{1}{2}A}{\cot\frac{1}{2}A - \tan\frac{1}{2}A};$$

$$\cot A = \frac{\cot^2\frac{1}{2}A - 1}{2\cot\frac{1}{2}A} = \frac{2 - \sec^2\frac{1}{2}A}{2\tan\frac{1}{2}A} = \frac{\operatorname{cosec}^2\frac{1}{2}A - 2}{2\cot\frac{1}{2}A};$$

$$\operatorname{cosec} A = \frac{1 + \tan^2\frac{1}{2}A}{2\tan\frac{1}{2}A} = \frac{\cot\frac{1}{2}A + \tan\frac{1}{2}A}{2} = \frac{\sec\frac{1}{2}A \operatorname{cosec}\frac{1}{2}A}{2}$$

(2) $\cos^4 A - \sin^4 A = \sin 2A.$

(3) $\tan(45^\circ + A) + \tan(45^\circ - A) = 2\sec 2A.$

(4) $\frac{1}{2}\tan(45^\circ + A) - \frac{1}{2}\tan(45^\circ - A) = \tan 2A.$

(5) $\dfrac{\cos 2A}{1 + \sin 2A} = \dfrac{1}{\tan 2A + \sec 2A}.$

(6) $\dfrac{\sin 2A}{\cot A} + \cos 2A = 1.$

(7) $4\sin^2(30^\circ + A) = 2\sin^2 A + \sqrt{3}.\sin 2A + 1.$

(8) $\cos 2A + \cos 2B = 2\cos(A + B)\cos(A - B).$

(9) $\cos\frac{1}{2}A = \dfrac{\tan A}{\sec A\sqrt{\operatorname{vers} A}}.$

(10) $8\cot 2A \operatorname{cosec}^2 A = \cot A \operatorname{cosec}^2 A - \tan A \sec^2 A.$

(11) $(\cos A + \sqrt{-1}\sin A - 1)(\cos A - \sqrt{-1} - 1)$

$$= 4\sin^2\tfrac{1}{2}A.$$

(12) Prove that $\dfrac{\sin\theta}{1+2\cos\theta} = \dfrac{1}{4}\cot\dfrac{\theta}{2} - \dfrac{3}{4}\cot\dfrac{3\theta}{2}$.

(13) Shew that $\tan 9^0 = 1 + \sqrt{5} - \sqrt{(5+2\sqrt{5})}$.

(14) Prove that $\tan(45^0 - x)\tan(45^0 - 3x) = \dfrac{1-2\sin 2x}{1+2\sin 2x}$.

(15) Eliminate θ from the equations
$\cos^3\theta + a\cos\theta = b$; $\sin^2\theta + a\sin\theta = c$.

(16) If $\tan\theta = \tan^3\dfrac{\phi}{2}$, and $\cos^2\phi = \dfrac{m^2-1}{3}$,

shew that $m = \dfrac{2}{\{(\cos\theta)^{\frac{2}{3}} + (\sin\theta)^{\frac{2}{3}}\}^{\frac{1}{2}}}$.

(17) $\tan(a+x)\tan(a-x) = \dfrac{1-2\cos 2a}{1+2\cos 2a}$, find $\sin x$,

$x = 30^0$.

(18) If $\dfrac{n\tan x}{\cos^2(a-x)} = \dfrac{m\tan(a-x)}{\cos^2 x}$, shew that

$$\tan(a-2x) = \frac{n-m}{n+m}\tan a.$$

(19) If $\left.\begin{aligned} 4x &= 3a\cos\theta + a\cos 3\theta \\ 4y &= 3a\sin\theta - a\sin 3\theta \end{aligned}\right\}$ shew that $(a^2 - x^2 - y^2)^3 = 27a^2x^2y^2$.

(20) If $\sin(\beta+x) + \sin(2\alpha - \beta + x)$
$= (\cos^2\beta - \cos^2\alpha)\sin(\alpha + 2x) - \sin\alpha(\sin^2\beta - \sin^2\alpha)$
shew that $\sec x = \sin(\alpha+\beta)\tan(\alpha - \beta)$.

(21) If $2\tan^{-1}x = \sin^{-1}2y$, then $y = \dfrac{x}{1+x^2}$.

(22) Given $\left.\begin{aligned} \cos\phi + \cos\theta &= a \\ \cos 5\phi + \cos 5\theta &= b \end{aligned}\right\}$ find $\cos\phi$ and $\cos\theta$.

(23) Given $\operatorname{vers}^{-1}(1+x) - \operatorname{vers}^{-1}(1-x)$
$= \tan^{-1} 2\sqrt{(1-x^2)}$, to find x,

$x = \pm 1, \dfrac{1}{2}$ and -1.

(24) $\tan^{-1}\left(\frac{1}{x-1}\right) - \tan^{-1}\left(\frac{1}{x+1}\right) = \frac{\pi}{2}$, find x,

$$x = \pm \frac{2}{\sqrt{3}-1}.$$

(25) $\sec^{-1} a - \sec^{-1} b = \sec^{-1} \frac{x}{b} - \sec^{-1} \frac{x}{a}$, find x,

$$x = \pm ab.$$

(26) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{1}{2y} - \tan^{-1} \frac{1}{2x} = \frac{\pi}{12}$,

find x and y.

(27) Prove that $A + \tan^{-1}(\cot 2A) = \tan^{-1}(\cot A)$.

(28) Find the sum of n terms of the series,

$$\tan^{-1} a + \tan^{-1} \frac{a}{1 + 1.2a^2} + \tan^{-1} \frac{a}{1 + 2.3a^2} + \&c.$$

(29) If $A + B + C = 180^0$, then

(1) $\sin^2 \frac{1}{2}A + \sin^2 \frac{1}{2}B + \sin^2 \frac{1}{2}C + 2 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C = 1$.

(2) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

(3) $\cos 2A + \cos 2B + \cos 2C = 1 - 4 \sin A \sin B \sin C$.

(30) If $A + B + C = 90^0$, then

(1) $\cos 2A + \cos 2B + \cos 2C = 1 + 4 \sin A \sin B \sin C$.

(2) $\tan A \tan B + \tan A \tan C + \tan B \tan C = 1$.

(3) $\tan A + \tan B + \tan C = \tan A \tan B \tan C + \sec A \sec B \sec C$.

CHAPTER III.

16. There are six parts in every plane triangle, viz., the three sides and the three angles, any three of which being given, the others can be found, except the case where the three angles are given; for it is clear that if you draw an indefinite number of lines parallel to the base AB of the triangle ABC, all the triangles so formed will have their

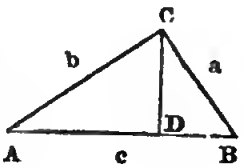
angles equal, but the triangles may have their sides of any length whatever (see fig.) The angles of a triangle are generally designated by A, B, C, and the sides opposite to them by the letters a, b, c.

The first proposition which we set out to prove is the following:

The sines of the angles of a plane triangle are proportional to the sides which subtend them; that is,

$$\sin A : \sin B :: a : b.$$

From any angle C of the triangle ABC, let fall the perpendicular CD, then, by the definitions,



$$\frac{CD}{AC} = \sin A, \text{ and } \frac{CD}{CB} = \sin B,$$

dividing the former by the latter, we have

$$\frac{\sin A}{\sin B} = \frac{CD}{AC} \cdot \frac{CB}{CD} = \frac{CB}{AC};$$

$$i.e.\ \sin A : \sin B :: CB : AC :: a : b.$$

In a similar way, by letting fall perpendiculars from A on BC, and B on AC, we have

$$\sin B : \sin C :: b : c,$$

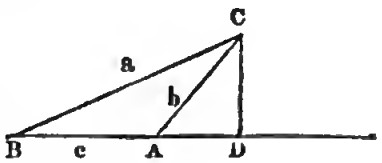
$$\sin A : \sin C :: a : c.$$

We now proceed to find the cosine of an angle in terms of the sides.

When the triangle is obtuse angled.

By Euclid, Book II. Prop. 12.

$$BC^2 = AB^2 + AC^2 + 2\,AB.AD \ldots (1).$$



Now $\frac{AD}{AC} = \cos CAD = -\cos BAC,$

because the cosine of an angle is equal to minus the cosine of its supplement;

$$\therefore\ AD = -AC \cos BAC,$$

this substituted in (1) we have

$$BC^2 = AB^2 + AC^2 - 2\,AB.AC \cos BAC,$$

$$a^2 = c^2 + b^2 - 2bc \cos A\ ;$$
$$\therefore\ \cos A = \frac{c^2 + b^2 - a^2}{2bc}.$$

When the triangle is acute angled, as in the first figure, we have, by Euclid, Book II. Prop. 13,

$$AC^2 = AB^2 + BC^2 - 2AB.DB \ldots (2).$$
$$\frac{DB}{BC} = \cos B,\ \therefore DB = BC \cos B\ ;$$

this substituted in (2) gives

$$AC^2 = AB^2 + BC^2 - 2AB\,.\,BC \cos B,$$
$$\text{or } b^2 = c^2 + a^2 - 2ac\,.\cos B\ ;$$
$$\therefore\ \cos B = \frac{a^2 + c^2 - b^2}{2ac}.$$

Now, by putting c for a, and a for c, and A for B, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

In the same manner

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

17. By p. 27, $\cos A = 1 - 2\sin^2 \frac{1}{2}A$,

$$\therefore\ 2\sin^2 \tfrac{1}{2}A = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc}$$
$$= \frac{2bc - b^2 - c^2 + a^2}{2bc}$$
$$= \frac{a^2 - (b^2 - 2bc + c^2)}{2bc} = \frac{a^2 - (b-c)^2}{2bc}.$$

Now, since the difference of the squares of any two quantities is equal to the product of the sum and difference of the same quantities, we have

$$2\sin^2 \tfrac{1}{2}A = \frac{(a + b - c)(a - b + c)}{2bc} \ldots\ldots (1).$$

Also, by p. 27, $\cos A = 2\cos^2 A - 1$;

$$\therefore\ 2\cos^2 \tfrac{1}{2}A = 1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc + b^2 + c^2 - a^2}{2bc}$$
$$= \frac{(b + c)^2 - a^2}{2bc} = \frac{(b + c + a)(b + c - a)}{2bc} \ldots\ldots (2).$$

Multiply (1) by (2),

$$4\sin^2\tfrac{1}{2}A\cos^2\tfrac{1}{2}A=\frac{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}{4b^2c^2}.$$

Extracting the square root,

$$2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A=\sqrt{\frac{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}{4b^2c^2}},$$

but $2\sin\frac{1}{2}A\cos\frac{1}{2}A=\sin A$;

$$\therefore\ \sin A=\frac{\sqrt{\{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)\}}}{2bc}\ldots(3).$$

This is generally put under a different form by putting the perimeter

$$a+b+c=2S.$$

From this equation subtract $2c$,

$$a+b-c=2S-2c=2(S-c).$$

From the same equation subtract $2b$,

$$a-b+c=2S-2b=2(S-b).$$

Also from the same subtract $2a$,

$$-a+b+c=2S-2a=2(S-a).$$

Substituting these values in equation (3),

$$\sin A=\frac{\sqrt{\{2S\,2(S-c)\,2(S-b)\,2(S-a)\}}}{2bc}$$

$$=\frac{\sqrt{\{16(S-a)(S-b)(S-c)\}}}{2bc}$$

$$=2\frac{\sqrt{\{S(S-a)(S-b)(S-c)\}}}{bc}\ldots\ldots(4).$$

This is sometimes done in the following manner:—

$$\sin^2 A=(1-\cos A)(1+\cos A)=\left(1-\frac{b^2+c^2-a^2}{2bc}\right)\left(1+\frac{b^2+c^2-a^2}{2bc}\right)$$

$$=\frac{(a+c-b)(a+b-c)(b+c+a)(b+c-a)}{4b^2c^2},$$

which is the same as equation (3).

From equation (1),

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(a+b-c)(a-b+c)}{4bc}} = \sqrt{\frac{(S-b)(S-c)}{bc}}.$$

From (2),

$$\cos \tfrac{1}{2}A = \sqrt{\frac{(b+c+a)(b+c-a)}{4bc}} = \sqrt{\frac{S(S-a)}{bc}};$$

$$\therefore \tan \tfrac{1}{2}A = \frac{\sin \frac{1}{2}A}{\cos \frac{1}{2}A} = \sqrt{\frac{(S-b)(S-c)}{S(S-a)}}.$$

These values being of great importance, we shall collect them for easy reference.

$$\sin A = \frac{2}{bc}\sqrt{\{S(S-a)(S-b)(S-c)\}} \ldots\ldots (a),$$

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(S-b)(S-c)}{bc}} \ldots\ldots\ldots\ldots (b),$$

$$\cos \tfrac{1}{2}A = \sqrt{\frac{S(S-a)}{bc}} \ldots\ldots\ldots\ldots\ldots (c),$$

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(S-b)(S-c)}{S(S-a)}} \ldots\ldots\ldots\ldots (d).$$

By the same method as above, $\sin B$ and $\sin C$ may be determined; but they may be more easily found from equation (4) thus, put B for A; a for b and b for a, and we have

$$\sin B = \frac{2}{ac}\sqrt{\{S(S-a)(S-b)(S-c)\}} \ldots\ldots (e).$$

Also putting c for A, a for c, and c for a, we have

$$\sin C = \frac{2}{ab}\sqrt{\{S(S-a)(S-b)(S-c)\}} \ldots\ldots (f).$$

PRACTICAL OBSERVATIONS.

Equation (a) may be used in all cases where A does not approach near to 90°.

Equation (b) may be used in all cases where A approaches near to 90°.

Equation (c) may also be used in all cases where A approaches near to 90°.

Equation (d) may be used in all cases where A does not approach near to 180^0.

18. The area of the triangle $ACB = \frac{AB.CD}{2}$,

but $\frac{CD}{AC} = \sin A$; $\therefore CD = AC \sin A$;

$$\therefore \Delta = \frac{AB.AC \sin A}{2} \ldots\ldots\ldots\ldots (5).$$

Where Δ represents the area of the triangle. We may also express the area of a triangle in terms of the sides from the above.

By equation (4),

$$\sin A = \frac{2}{bc}\sqrt{\{S(S-a)(S-b)(S-c)\}}.$$

This substituted in equation (5) gives

$$\Delta = \frac{1}{2} AB.AC.\frac{2}{bc}\sqrt{\{S(S-a)(S-b)(S-c)\}}$$

$$= \frac{1}{2}bc.\frac{2}{bc}\sqrt{\{S(S-a)(S-b)(S-c)\}}$$

$$= \sqrt{\{S(S-a)(S-b)(S-c)\}}.$$

19. When two sides and their included angle are given,

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

Add unity to each side of this equation,

$$\frac{a}{b} + 1 = \frac{\sin A}{\sin B} + 1; \quad \therefore \frac{a+b}{b} = \frac{\sin A + \sin B}{\sin B} \ldots (1).$$

Subtract unity from each side of the same equation,

$$\frac{a}{b} - 1 = \frac{\sin A}{\sin B} - 1; \quad \therefore \frac{a-b}{b} = \frac{\sin A - \sin B}{\sin B} \ldots (2).$$

Divide (1) by (2) and we have

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}.$$

But by Art. 17, $\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$, and $\sin A - \sin B = 2 \sin \frac{1}{2}(A-B) \cos \frac{1}{2}(A+B)$;

$$\therefore \frac{a+b}{a-b}=\frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)}=\tan\tfrac{1}{2}(A+B)\cot\tfrac{1}{2}(A-B)$$

$$=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)},$$

since $\tan\frac{1}{2}(A+B)=\cot\frac{1}{2}C$.

In logarithms,

$$\log\tan\tfrac{1}{2}(A-B)=\log(a-b)-\log(a+b)+\log\cot\tfrac{1}{2}C.$$

From $\frac{c}{a}=\frac{\sin C}{\sin A}$, we have

$$\log c=\log a+\log\sin C-\log\sin A;$$

c may also be found from the formula,

$\cos C=\frac{a^2+b^2-c^2}{2ab}$; which may be adapted to logarithms in the following manner:

$$c^2=a^2+b^2-2ab\cos C;$$

$$\text{but } \cos C=1-2\sin^2\tfrac{1}{2}C;$$

$$\therefore c^2=a^2+b^2-2ab(1-2\sin^2\tfrac{1}{2}C)$$

$$=a^2+b^2-2ab+4ab\sin^2\tfrac{1}{2}C$$

$$=(a-b)^2+4ab\sin^2\tfrac{1}{2}C$$

$$=(a-b)^2\{1+\frac{4ab}{(a-b)^2}\sin^2\tfrac{1}{2}C\};$$

$$\text{now since } \frac{4ab}{(a-b)^2}\sin^2\tfrac{1}{2}C=\left\{\frac{2a^{\frac{1}{2}}b^{\frac{1}{2}}\sin\frac{1}{2}C}{a-b}\right\}^2,$$

which, being a square, is necessarily a positive quantity, and as this may be of any magnitude whatever, we may put

$$\tan^2\theta=\frac{4ab}{(a-b)^2}\sin^2\tfrac{1}{2}C;$$

$$\text{then } c^2=(a-b)^2(1+\tan^2\theta)=(a-b)^2\sec^2\theta;$$

$$\therefore c=(a-b)\sec\theta,$$

which is adapted for logarithmic computation, thus:

$$\log c=\log(a-b)+\log\sec\theta-10.$$

The following is useful in cases where a and b are nearly equal:

$$c^2=a^2+b^2-2ab\cos C;$$

$$\text{but } \cos C=2\cos^2\tfrac{1}{2}C-1;$$

$$\therefore\ c^2 = a^2 + b^2 - 2ab\,(2\cos^2\tfrac{1}{2}C - 1)$$
$$= a^2 + b^2 + 2ab - 4ab\,\cos^2\tfrac{1}{2}C$$
$$= (a+b)^2 - 4ab\,\cos^2\tfrac{1}{2}C = (a+b)^2\left\{1 - \frac{4ab\,\cos^2\frac{1}{2}C}{(a+b)^2}\right\}.$$

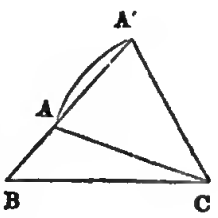
Now $\dfrac{4ab\,\cos^2\frac{1}{2}C}{(a+b)^2}$ is always less than unity,

for $\sqrt{ab}$ is less than $\frac{1}{2}(a+b)$*, or $(a+b)^2$ is greater than $4ab$, and since the cosine cannot exceed unity, it is evident that the above is a proper fraction, and we may put

$$\sin^2\theta = \frac{4ab\,\cos^2\frac{1}{2}C}{(a+b)^2};$$
$$\therefore\ c^2 = (a+b)^2\,(1 - \sin^2\theta) = (a+b)^2\cos^2\theta,$$
$$\text{or } c = (a+b)\cos\theta,$$
$$\log c = \log(a+b) + \log\cos\theta - 10;$$
$$\text{since } \tan\theta = \frac{2a^{\frac{1}{2}}b^{\frac{1}{2}}}{a-b}\sin\tfrac{1}{2}C,$$
$$\log\tan\theta = \log 2 + \frac{1}{2}\log a + \frac{1}{2}\log b + \log\sin\tfrac{1}{2}C - \log(a-b).$$
$$\text{Also, since } \sin\theta = \frac{2a^{\frac{1}{2}}b^{\frac{1}{2}}}{a+b}\cos\tfrac{1}{2}C,$$
$$\log\sin\theta = \log 2 + \frac{1}{2}\log a + \frac{1}{2}\log b + \log\cos\tfrac{1}{2}C - \log(a+b).$$

ON THE AMBIGUOUS CASE.

20. When two sides of a triangle and the angle opposite one of them are given, there is evidently an ambiguity; for if CA be taken less than BC, but greater than a perpendicular from C on BA or BA produced, and if with C as centre and CA as radius, a circle be



* $\dfrac{a+b}{2} > \sqrt{ab}$ or $a+b > 2\sqrt{ab}$ or $a^2 + 2ab + b^2 > 4ab$,
or $a^2 - 2ab + b^2 > 0$, or $(a-b)^2 > 0$;
for any even power of a quantity is positive.

described, it must necessarily cut the line BAA' in two points, while in all other cases it will either fall short of it or touch it only in one point. When AC is less than $a \sin B$, the triangle is impossible: but when $AC = a \sin B$, the triangle is right angled, and there is no ambiguity. And when AC is greater than BC, there can be only one triangle, and, if it fall between these limits, there is an ambiguity, for there will be two triangles having the same data; thus when the least side AC is opposite the given angle B, it is evident that either ABC or $A'BC$ may be the triangle required.

21. To find the area of a triangle in the terms of the radius of the inscribed circle, and sides of the triangle.

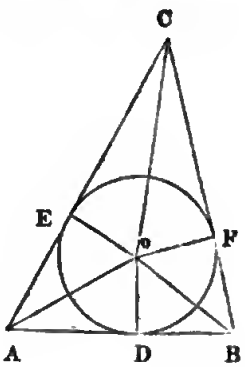
The triangle ACB is composed of the triangles AOB, AOC, and BOC; O being the centre of the inscribed circle.

Now the area of the triangle

$$AOB = \frac{AB \cdot OD}{2} = \frac{cr}{2},$$

$$AOC = \frac{AC \cdot OE}{2} = \frac{br}{2},$$

$$BOC = \frac{BC \cdot OF}{2} = \frac{ar}{2}.$$



Hence the area of the whole triangle

$$ACB = \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2} = \frac{a+b+c}{2} \cdot r.$$

Since $\frac{a+b+c}{2} = S$, the semi-perimeter, we may express the radius in terms of the sides; putting, as before, Δ for the area of the triangle,

$$\Delta = rS; \quad \therefore\ r = \frac{\Delta}{S} \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

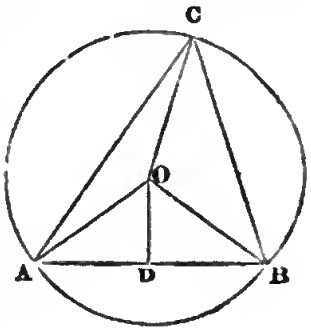
but $\Delta = \sqrt{\{S(S-a)(S-b)(S-c)\}}$ by p. 56;

$$\therefore\ r = \frac{\sqrt{\{S(S-a)(S-b)(S-c)\}}}{S}$$

$$= \sqrt{\frac{(S-a)(S-b)(S-c)}{S}} \ldots\ldots\ldots (2).$$

22. To find the area of the triangle in terms of the radius of the circumscribing triangle and the sides.

By the 20th prop. of the third book of Euclid, the angle AOB is double the angle ACB.



Draw OD perpendicular to AB, which will bisect the angle AOB and the side AB.

Now by p. 56,

$$\Delta = \frac{AC.CB\sin ACB}{2} = \frac{AC.CB\sin AOD}{2}$$

$$= \frac{ba}{2}\cdot\frac{AD}{OA} = \frac{ba}{2}\cdot\frac{1}{2}\cdot\frac{AB}{OA} = \frac{abc}{4R}.$$

We may now find the radius of the circumscribing circle in terms of the sides; for since

$$\Delta = \frac{abc}{4R},\ \therefore R = \frac{abc}{4\Delta} = \frac{abc}{4\sqrt{\{S(S-a)(S-b)(S-c)\}}} \cdots (3).$$

Multiplying (2) by (3),

$$Rr = \frac{abc}{4S} = \frac{abc}{2(a+b+c)}.$$

By dividing (3) by (2),

$$\frac{R}{r} = \frac{abc}{4(S-a)(S-b)(S-c)}.$$

Examples.

(1) In a plane triangle if a, b, c be the sides, and A, B, C the angles, prove that

$$c = b\cos A \pm \sqrt{(a^2 - b^2\sin^2 A)}.$$

$$\text{Since } \cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$2bc\cos A = b^2 + c^2 - a^2,$$

$$c^2 - 2bc\cos A = a^2 - b^2,$$

$$c^2 - 2bc\cos A + b^2\cos^2 A = a^2 - b^2 + b^2\cos^2 A$$

$$= a^2 - (1 - \cos^2 A)\, b^2,$$

$$c^2 - 2bc\cos A + b^2\cos^2 A = a^2 - b^2\sin^2 A,$$
$$c - b\cos A \quad \sqrt{(a^2 - b^2\sin^2 A)},$$
$$c = b\cos A \pm \sqrt{(a^2 - b^2\sin^2 A)}.$$

(2) Find the area of a square when the difference between the diagonal and the side $= m$ feet.

Let x = side of the square, then $x + m$ = diagonal,

$$x^2 + x^2 = (x + m)^2, \text{ or } 2x^2 = (x + m)^2,$$
$$x\sqrt{2} = x + m,$$
$$x\sqrt{2} - x = m,$$
$$x(\sqrt{2} - 1) = m,$$
$$x = \frac{m}{\sqrt{2} - 1} = (\sqrt{2} + 1)\,m,$$
$$x^2 = m^2(\sqrt{2} + 1)^2.$$

(3) Given the area, angle C, and $a + b$; find the sides of the triangle

$$ab\sin C = 2\Delta,$$
$$ab = \frac{2\Delta}{\sin C}.$$

Let $a + b = m$,

$$a^2 + 2ab + b^2 = m^2,$$
$$4ab = \frac{8\Delta}{\sin C};$$
$$\therefore\ a^2 - 2ab + b^2 = m^2 - \frac{8\Delta}{\sin C},$$
$$a - b = \sqrt{\left(m^2 - \frac{8\Delta}{\sin C}\right)},$$

but $a + b = m$;

$$\therefore\ a = \frac{m + \sqrt{\left(m^2 - \frac{8\Delta}{\sin C}\right)}}{2},$$
$$b = \frac{m - \sqrt{\left(m^2 - \frac{8\Delta}{\sin C}\right)}}{2},$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

whence c may be found; or c may be found by the formula

$$c^2 = (a+b)^2 - 4ab\cos^2\frac{C}{2}. \quad \text{(p. 58).}$$

(4) Given c, the base of an isosceles triangle, and p the perpendicular from one of the equal angles upon the opposite side: shew that the area $= \dfrac{pc^2}{4\sqrt{(c^2-p^2)}}$.

By the 13th Prop. of the second book of Euclid

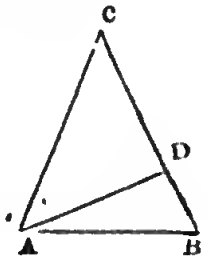
$$AC^2 = c^2 + BC^2 - 2BC.BD,$$

but $AC = BC$;

$$\therefore\ BC^2 = c^2 + BC^2 - 2BC.BD;$$

$$\therefore\ BC = \frac{c^2}{2BD} = \frac{c^2}{2\sqrt{(c^2-p^2)}},$$

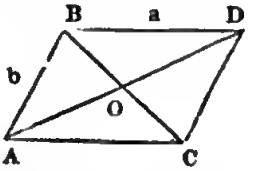
$$\text{area} = \frac{BC.AD}{2} = \frac{pc^2}{4\sqrt{(c^2-p^2)}}.$$



(5) If θ be the angle between the diagonals of a parallelogram whose sides a and b are inclined to each other at an angle α,

$$\text{then } \tan\theta = \frac{2ab\sin\alpha}{a^2-b^2}.$$

Let $DB = a$, and $AB = b$,



then $a^2 = OB^2 + OD^2 + 2BO.OD\cos\theta$,

$$b^2 = OB^2 + AO^2 - 2AO.OB\cos\theta,$$

$$a^2 - b^2 = 2AO.OB\cos\theta + 2BO.OD\cos\theta$$

$$= 2BO(AO+OD)\cos\theta = 2BO.AD\cos\theta,$$

area of parallelogram $= ab\sin\alpha$; but $AD.BO\sin\theta$ is also equal the area, hence by division

$$\frac{ab\sin\alpha}{a^2-b^2} = \frac{AD.BO\sin\theta}{2BO.AD\cos\theta} = \frac{1}{2}\tan\theta,$$

$$\therefore\ \tan\theta = \frac{2ab\sin\alpha}{a^2-b^2}.$$

(6) In a triangle ABC prove that

$$\frac{\sin(A-B)}{\sin C} = \frac{a^2-b^2}{c^2},$$

$$\frac{\sin(A-B)}{\sin C} = \frac{\sin A\cos B - \cos A\sin B}{\sin C}$$

$$= \frac{\sin A}{\sin C}\cos B - \frac{\sin B}{\sin C}\cos A$$

$$= \frac{a}{c}\cdot\frac{a^2+c^2-b^2}{2ac} - \frac{b}{c}\cdot\frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{a^2+c^2-b^2}{2c^2} - \frac{b^2+c^2-a^2}{2c^2}$$

$$= \frac{a^2-b^2}{c^2}.$$

(7) Prove that the area of a triangle ABC

$$= \frac{a^2+b^2+c^2}{4(\cot A+\cot B+\cot C)}.$$

Since $\cos A = \frac{b^2+c^2-a^2}{2bc}$, and by p. 56, $\sin A = \frac{2\Delta}{bc}$

$$\cos B = \frac{a^2+c^2-b^2}{2ac}, \qquad \sin B = \frac{2\Delta}{ac};$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}, \qquad \sin C = \frac{2\Delta}{ab}.$$

By dividing each in the first group by the corresponding one in the second, we have

$$\cot A = \frac{b^2+c^2-a^2}{4\Delta}, \quad \cot B = \frac{a^2+c^2-b^2}{4\Delta}, \quad \cot C = \frac{a^2+b^2-c^2}{4\Delta};$$

$\therefore$ by addition,

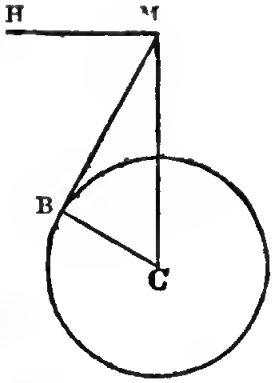
$$\cot A+\cot B+\cot C = \frac{b^2+c^2-a^2+a^2+c^2-b^2+a^2+b^2-c^2}{4\Delta}$$

$$= \frac{a^2+b^2+c^2}{4\Delta};$$

$$\therefore\ \Delta = \frac{a^2+b^2+c^2}{4(\cot A+\cot B+\cot C)}.$$

(8) From the top of a mountain a miles high, the visible horizon appeared depressed $A°$; required the diameter ($2r$) of the earth, and the distance (d) of the horizon.

In the annexed figure $CM = r + a$,



and since $HMB = A$, BCM is evidently $= A$,

$$CM \cos A = BC \text{ or } (r + a) \cos A = r,$$

$$\therefore\ r = \frac{a \cos A}{1 - \cos A} \quad \frac{a \cot A}{\dfrac{1 - \cos A}{\sin A}};$$

(Dividing numerator and denominator by $\sin A$)

$$= \frac{a \cot A \sin A}{2 \sin^2 \frac{1}{2} A} = \frac{a \cot A \,.\, 2 \sin \frac{1}{2} A \cos \frac{1}{2} A}{2 \sin^2 \frac{1}{2} A}$$

$$= a \cot A \cot \tfrac{1}{2} A;$$

$$\frac{BC}{BM} = \cot A, \text{ or } \frac{r}{d} = \cot A;\ \therefore\ d = \frac{r}{\cot A} = \frac{a \cot A \cot \frac{1}{2} A}{\cot A}$$

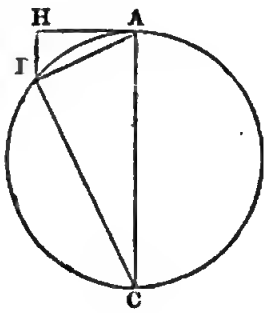
$$= a \cot \tfrac{1}{2} A.$$

(9) If L = length in miles of an arc of a great circle of the earth, and D the depression in feet of one extremity of it below a tangent to the other, then $D = \frac{2}{3} L^2$ nearly.

$AC = 7920$ miles nearly, $AB = L$, $HB = D$ feet $= \dfrac{D}{5280}$ miles,

$\because$ the triangles AHB and ACB are similar;

$$\therefore\ HB : L :: L : AC,$$



$$\frac{D}{5280} : L :: L : 7920,$$

$$D \,.\, \frac{7920}{5280} = L^2.$$

$$D \,.\, \frac{3}{2} = L^2,$$

$$\therefore\ D = \frac{2L^2}{3} \text{ nearly.}$$

(10) Given the perimeter, and the area of the triangle, and the angle A; find a.

By p. 55, $\cos \frac{1}{2} A = \sqrt{\dfrac{S(S-a)}{bc}}$,

$$bc \sin A = 2\Delta, \quad \therefore\ bc = \frac{2\Delta}{\sin A},$$

$$\cos^2 \tfrac{1}{2}A = \frac{S(S-a)}{2\Delta} \cdot \sin A = \frac{S(S-a)}{2\Delta} \cdot 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A.$$

Divide by $\sin \frac{1}{2}A \cos \frac{1}{2}A$, and

$$\cot \tfrac{1}{2}A = \frac{S^2 - Sa}{\Delta}, \text{ or } Sa = S^2 - \Delta \cot \tfrac{1}{2}A;$$

$$\therefore\ a = \frac{S^2 - \Delta \cot \frac{1}{2}a}{S}.$$

(11) Given the area, side c, and $A + B$; find $A - B$.

$$\text{Since } \cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

$$a^2 + b^2 - c^2 = 2ab \cos C,$$

$$a^2 + b^2 = c^2 + 2ab \cos C,$$

$$2ab = \frac{4\Delta}{\sin C},$$

$$a^2 + 2ab + b^2 = c^2 + 2ab \cos C + \frac{4\Delta}{\sin C}$$

$$= c^2 + \frac{4\Delta}{\sin C} \cos C + \frac{4\Delta}{\sin C}$$

$$= c^2 + 4\Delta \{\cot C + \sqrt{(1 + \cot^2 C)}\},$$

$$a + b = \sqrt{[c^2 + 4\Delta \{\cot C + \sqrt{(1 + \cot^2 C)}\}]}.$$

In the same way

$$a - b = \sqrt{[c^2 + 4\Delta \{\cot C - \sqrt{(1 + \cot^2 C)}\}]},$$

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}A + B}{\tan \frac{1}{2}A - B} = \sqrt{\frac{c^2 + 4\Delta \{\cot C + \sqrt{(1 + \cot^2 C)}\}}{c^2 + 4\Delta \{\cot C - \sqrt{(1 + \cot^2 C)}\}}},$$

$$\therefore \tan \tfrac{1}{2}(A-B) = \tan \tfrac{1}{2}(A+B) \sqrt{\frac{c^2 + 4\Delta \{\cot C - \sqrt{(1 + \cot^2 C)}\}}{c^2 + 4\Delta \{\cot C + \sqrt{(1 + \cot^2 C)}\}}}.$$

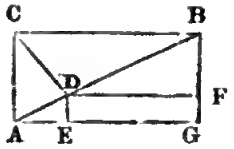
(12) If from one of the angles of a rectangle a perpendicular be drawn to its diagonal, and from their intersection, lines be drawn perpendicular to the sides containing the opposite angle, then putting P and p for these last perpendiculars, and D for the diagonal, $P^{\frac{2}{3}} + p^{\frac{2}{3}} = D^{\frac{2}{3}}$.

Let the angle $BAG = \theta$, then $AD \sin \theta = DE = P$;

but by (Euc. VI. 8th Cor.) $AB \cdot AD = AC^2$, $\therefore\ AD = \frac{AC^2}{AB}$;

but $AC = AB \sin\theta$;

$$\therefore AD = \frac{AB^2 \sin^2\theta}{AB} = AB \sin^2\theta;$$



and since $P = AD \sin\theta$, we have

$$P = AB \sin^3\theta = D \sin^3\theta.$$

In the same way we have

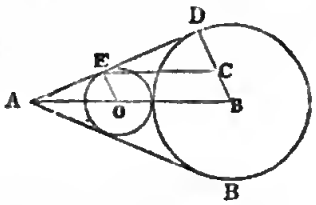
$$p = D \cos^3\theta,$$

$$\therefore P^{\frac{2}{3}} = D^{\frac{2}{3}} \sin^2\theta, \text{ and } p^{\frac{2}{3}} = D^{\frac{2}{3}} \cos^2\theta.$$

By addition,

$$P^{\frac{2}{3}} + p^{\frac{2}{3}} = D^{\frac{2}{3}} (\sin^2\theta + \cos^2\theta) = D^{\frac{2}{3}}.$$

(13) If two circles, whose radii are a and b, touch each other externally, and if θ be the angle contained by the two common tangents to these circles, shew that



$$\sin\theta = \frac{4(a-b)\sqrt{ab}}{(a+b)^2}.$$

Let $AB = x$, $OB = EC = a + b$, $CD = BD - EO = a - b$;

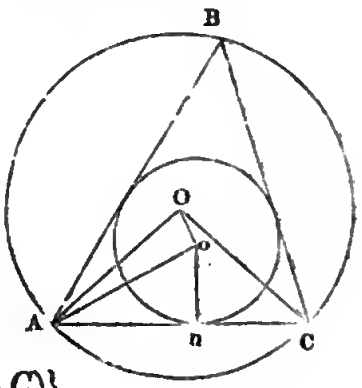
$$\sin\tfrac{1}{2}\theta = \frac{BD}{AB} = \frac{DC}{EC} = \frac{a-b}{a+b},$$

$$\cos\tfrac{1}{2}\theta = \sqrt{(1 - \sin^2\tfrac{1}{2}\theta)} = \sqrt{\left\{1 - \left(\frac{a-b}{a+b}\right)^2\right\}} = \frac{2\sqrt{ab}}{a+b},$$

$$\sin\theta = 2 \sin\tfrac{1}{2}\theta \cos\tfrac{1}{2}\theta = \frac{4(a-b)}{(a+b)^2}\sqrt{ab}.$$

(14) To find the distance between the centres of the inscribed and circumscribed circles, the figure being the triangle ABC.

Let O and o be the centres of circumscribed and inscribed circles,



$$Oo = \delta, \quad OA = R, \quad on = r,$$

$$\therefore Ao = \frac{r}{\sin\frac{1}{2}A},$$

$$\angle OAo = OAC - oAc$$

$$= \tfrac{1}{2}(180^\circ - AOC) - \tfrac{1}{2}A$$

$$= 90^\circ - B - \tfrac{1}{2}A$$

$$= 90^\circ - B - \{90^\circ - \tfrac{1}{2}(B + C)\}$$

$$= \tfrac{1}{2}(C - B);$$

$$\delta^2 = Ao^2 + Ao^2 - 2\,AoAo\,.\cos OAo$$

$$= R^2 + \frac{r^2}{\sin^2 \frac{1}{2}A} - \frac{2Rr}{\sin \frac{1}{2}A}\cos \tfrac{1}{2}(C - B);$$

$$\therefore (\delta^2 - R^2)\sin^2 \tfrac{1}{2}A = r^2 - 2Rr\,.\sin \tfrac{1}{2}A \cos \tfrac{1}{2}(C - B)$$

$$= r^2 - 2Rr\cos \tfrac{1}{2}(B + C)\cos \tfrac{1}{2}(C - B);$$

$$\therefore (\delta^2 - R^2)(1 - \cos A) = 2r^2 - 2Rr\,(\cos B + \cos C).$$

Similarly,

$$(\delta^2 - R^2)(1 - \cos C) = 2r^2 - 2Rr\,(\cos B + \cos A).$$

Subtract

$$(\delta^2 - R^2)(\cos C - \cos A) = -2Rr\,.\,(\cos C - \cos A),$$

$$\therefore \delta^2 - R^2 = -2Rr;$$

$$\therefore \delta^2 = R^2 - 2Rr; \qquad \therefore \delta = \sqrt{(R^2 - 2Rr)}.$$

(15) If the angles of a triangle be in geometrical proportion, ratio $\frac{1}{2}$, the greatest side

$$= 2 \text{ perimeter} \times \sin 12^\circ 51' 25'' \tfrac{5}{7}.$$

Let $P = a + b + c$, a = greatest side,

$$A + \tfrac{1}{2}A + \tfrac{1}{4}A = 180^\circ.$$

$$\tfrac{1}{2}A = B, \ \tfrac{1}{4}A = C,$$

$$A = 102^\circ 51' 25'' \tfrac{5}{7},$$

$$B = \ 51^\circ 25' 42'' \tfrac{6}{7},$$

$$C = \ 25^\circ 42' 51'' \tfrac{3}{7};$$

$$P = a + b + c = a\left\{1 + \frac{b}{a} + \frac{c}{a}\right\} = a\left\{1 + \frac{\sin B}{\sin A} + \frac{\sin C}{\sin A}\right\}$$

$$= a\,\frac{\sin A + \sin B + \sin C}{\sin A};$$

$$\therefore a = \frac{P \sin A}{\sin A + \sin B + \sin C}$$

$$= \frac{2P \sin \frac{1}{2}A \cos \frac{1}{2}A}{4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C} = \frac{P \sin \frac{1}{2}A}{2 \cos \frac{1}{2}B \cos \frac{1}{2}C}.$$

But $\frac{1}{2}A = B$, $\frac{1}{2}B = C$,

$$a = \frac{P \sin B}{2 \cos C \cos \frac{1}{2}C} = \frac{2P \sin \frac{1}{2}B \cos \frac{1}{2}B}{2 \cos C \cos \frac{1}{2}C} = \frac{P \sin C \cos C}{\cos C \cos \frac{1}{2}C}$$

$$= \frac{P \sin C}{\cos \frac{1}{2}C} = \frac{2P \sin \frac{1}{2}C \cos \frac{1}{2}C}{\cos \frac{1}{2}C} = 2P \sin \tfrac{1}{2}C,$$

$= 2$ perimeter $\times \sin 12^\circ 51' 25'' \frac{5}{7}$. Answer.

(16) The sides of a triangle are in arithmetical proportion, its area $= \frac{3}{5}$ of the area of an equilateral triangle, having the same perimeter, shew that the greatest angle $= 120^\circ$, and that the sides are as $3 : 5 : 7$.

Let $a - x$, a, $a + x$ = the sides, then $3a$ = perimeter;

$\therefore$ a = side of equilateral Δ, and $\frac{a}{2}$ = half its base,

$$\sqrt{\left(a^2 - \frac{a^2}{4}\right)} = \sqrt{\frac{4a^2 - a^2}{4}} = \frac{a\sqrt{3}}{2} = \text{the altitude} = \frac{a}{2}\sqrt{3}.$$

Now $\because$ both Δ's may have the common base a; $\therefore$ area equilateral Δ : area of the other :: altitude of equilateral Δ : altitude of the other;

$$\text{or, } 1 : \frac{3}{5} :: \frac{a}{2}\sqrt{3} : \frac{3a}{10}\sqrt{3} = AD,$$

$$\sqrt{\left\{(a+x)^2 - \frac{27a^2}{100}\right\}} - \sqrt{\left\{(a-x)^2 - \frac{27}{100}a^2\right\}} = a,$$

$$\sqrt{\left\{(a+x)^2 - \frac{27}{100}a^2\right\}} = a + \sqrt{\left\{(a-x)^2 - \frac{27}{100}a^2\right\}};$$

$$\therefore (a+x)^2 - \frac{27}{100}a^2$$

$$= a^2 + 2a\sqrt{\left\{(a-x)^2 - \frac{27}{100}a^2\right\}} + (a-x)^2 - \frac{27}{100}a^2;$$

$$\text{or, } 2ax = a^2 + 2a\sqrt{\left\{(a-x)^2 - \frac{27}{100}a^2\right\}} - 2ax,$$

$$4ax = a^2 + 2a\sqrt{\left\{(a-x)^2 - \frac{27}{100}a^2\right\}},$$

$$4x - a = 2\sqrt{\{(a-x)^2 - \frac{27}{100}a^2\}};$$

$$16x^2 - 8ax + a^2 = 4(a-x)^2 - \frac{27}{25}a^2$$

$$= 4a^2 - 8ax + 4x^2 - \frac{27}{25}a^2,$$

$$12x^2 = 3a^2 - \frac{27}{25}a^2,$$

$$4x^2 = a^2 - \frac{9}{25}a^2,$$

$$4x^2 = \frac{16a^2}{25},$$

$$x^2 = \frac{4a^2}{25}; \quad \therefore\ x = \frac{2}{5}a,$$

$$a - x = \frac{3}{5}a,$$

$$a + x = \frac{7}{5}a,$$

$$AC = 3, \quad BC = 5, \quad AB = 7,$$

$\therefore \frac{3a}{5}$, a, $\frac{7a}{5}$ are the three sides, which are as $3a$, $5a$, $7a$, or as 3, 5, 7;

$$\cos C = \frac{AC^2 + BC^2 - AB^2}{2AC.BC} = \frac{3^2 + 5^2 - 7^2}{2 \times 3 \times 5} = -\frac{15}{30} = -\frac{1}{2},$$

but $\cos 120^\circ = -\frac{1}{2}$, therefore $C = 120^\circ$.

(17) The angles of a triangle are as the numbers 1, 2, 3; and the perpendicular from the greatest angle upon the opposite side is p. Shew that the area $= \frac{2p^2}{\sqrt{3}}$,

$$A + 2A + 3A = 180^\circ;$$

$$\therefore\ A = 30^\circ; \quad B = 60^\circ; \quad C = 90^\circ.$$

$$\sin A : \sin ACD :: p : AD, \quad \text{(fig. p. 52)}$$

$$AD = p\,\frac{\sin ACD}{\sin A} = p\,\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = p\sqrt{3};$$

$$\sin B : \sin BCD :: p : DB,$$

$$DB = p\,\frac{\sin BCD}{\sin B} = p\,\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{p}{\sqrt{3}},$$

$$AD + DB = AB = p\sqrt{3} + \frac{p}{\sqrt{3}} = \frac{3p+p}{\sqrt{3}} = \frac{4p}{\sqrt{3}},$$

$$\text{but area} = \frac{AB \times p}{2} = \frac{4p}{\sqrt{3}} \times \frac{p}{2} = \frac{2p^2}{\sqrt{3}}.$$

(18) The angles of a triangle are as the numbers 2, 3, 5, and the side opposite the greatest angle = 100 feet; find the remaining sides and the angles:

$$2A + 3A + 5A = 180^0;$$

$$\therefore \left.\begin{aligned} A &= 36^0 \\ B &= 54^0 \\ C &= 90^0 \end{aligned}\right\} \text{the angles,}$$

$$\sin C : \sin B :: c : b,$$

$$b = c\,\frac{\sin B}{\sin C},$$

$$\log b = \log c + \log \sin B - \log \sin C,$$

$$\log c = \log 100 \quad = 2{\cdot}00000$$

$$\log \sin B = \log \sin 54^0 = 9{\cdot}90796$$

$$11{\cdot}90796$$

$$\log \sin C = \log \sin 90^0 = 10{\cdot}00000$$

$$1{\cdot}90796 = \log 80{\cdot}9.$$

$$\sin C : \sin A :: c : a,$$

$$a = c\,\frac{\sin A}{\sin C}.$$

$$\log a = \log c + \log \sin A - \log \sin C$$
$$\log c = \log 100 = \quad 2{\cdot}00000$$
$$\log \sin A = \log \sin 36^0 = 9{\cdot}76922$$
$$11{\cdot}76922$$
$$\log \sin C = \log \sin 90^0 = 10{\cdot}00000$$
$$1{\cdot}76922 = \log 58{\cdot}78$$

$$\left.\begin{aligned} a &= 58{\cdot}78 \\ b &= 80{\cdot}9 \end{aligned}\right\} \text{sides.}$$

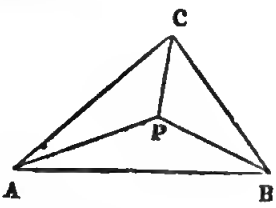
(19) Find a point within a triangle at which the three sides shall subtend equal angles:

$$\left.\begin{aligned} BC &= a \\ AC &= b \\ AB &= c \end{aligned}\right\} \qquad \left.\begin{aligned} AP &= x \\ BP &= y \\ PC &= z \end{aligned}\right\}.$$

Then $\angle APB = BPC = APC = 120^0$;

$\therefore$ the cosine of each $= -\frac{1}{2}$, and the sine of each $= \frac{\sqrt{3}}{2}$;

$$a^2 = BP^2 + PC^2 - 2\,BP \,.\, PC \cos 120^0$$
$$= BP^2 + PC^2 - 2\,BP \,.\, PC\left(-\frac{1}{2}\right)$$
$$= BP^2 + PC^2 + BP \,.\, PC,$$
$$b^2 = AP^2 + PC^2 + AP \,.\, PC,$$
$$c^2 = AP^2 + BP^2 + AP \,.\, PB$$



$$a^2 + b^2 + c^2 = 2x^2 + 2y^2 + 2z^2 + xy + yz + xz.$$

Let $\Delta ABC = m^2$,

$$m^2 = \Delta BPC + \Delta APB + \Delta APC;$$

$$\therefore\ m^2 = \frac{yz \sin 120^0}{2} + \frac{xy \sin 120^0}{2} + \frac{xz \sin 120^0}{2}$$

$$= \frac{\sqrt{3}}{2}\ (xy + yz + xz),$$

$$xy + yz + xz = \frac{4m^2}{\sqrt{3}},$$

$$3xy + 3yz + 3xz = 4m^2\sqrt{3}.$$

But $2(x^2 + y^2 + z^2) + (xy + yz + xz) = a^2 + b^2 + c^2$;

$$\therefore\ 2(x^2 + y^2 + z^2) + 4(xy + yz + xz) = a^2 + b^2 + c^2 + 4m^2\sqrt{3},$$

$$x^2+y^2+z^2+2(xy+yz+xz)=\left(\frac{a^2+b^2+c^2}{2}+2m^2\sqrt{3}\right),$$

$$x+y+z=\sqrt{\left(\frac{a^2+b^2+c^2}{2}+2m^2\sqrt{3}\right)}=\beta,$$

$$x+y+z=\beta,$$

$$x+y=\beta-z,$$

$$x^2+2xy+y^2=(\beta-z)^2, \text{ but } x^2+y^2+xy=c^2;$$

$$\therefore\ xy=(\beta-z)^2-c^2.$$

Also, since $xy+xz+yz=\frac{4m^2}{\sqrt{3}}$,

$$xy=\frac{4m^2}{\sqrt{3}}-z(y+x);\ \text{ but } x+y+z=\beta;$$

$$\therefore\ xy=\frac{4m^2}{\sqrt{3}}-z(\beta-z);$$

$$\therefore\ (\beta-z)^2-c^2+z(\beta-z)=\frac{4m^2}{\sqrt{3}};$$

$$\text{or } (\beta-z+z)(\beta-z)-c^2=\frac{4m^2}{\sqrt{3}};$$

$$\therefore\ \beta^2-\beta z-c^2=\frac{4m^2}{\sqrt{3}},$$

$$\beta z=\beta^2-c^2-\frac{4m^2}{\sqrt{3}}=\frac{a^2+b^2+c^2}{2}+2m^2\sqrt{3}-c^2-\frac{4m^2}{\sqrt{3}}$$

$$=\frac{a^2+b^2-c^2}{2}+\frac{2m^2}{\sqrt{3}};$$

$$\therefore\ z=\frac{1}{\beta}\left(\frac{a^2+b^2-c^2}{2}+\frac{2m^2}{\sqrt{3}}\right).$$

$$\text{Also } x=\frac{1}{\beta}\left(\frac{b^2+c^2-a^2}{2}+\frac{2m^2}{\sqrt{3}}\right),$$

$$\text{and } y=\frac{1}{\beta}\left(\frac{a^2+c^2-b^2}{2}+\frac{2m^2}{\sqrt{3}}\right).$$

(20) An object 6 feet high, placed on the top of a tower, subtends an angle = $\tan^{-1}\cdot015$, at a place whose horizontal distance from the foot of the tower is 100 feet, determine the tower's height.

Let $DAB = a$, $AD = x$, $AB = y$,

then $\tan a = \cdot 015$, $\therefore\ a = 51' 33''$;

$\therefore\ \sin a = \cdot 01513$, $\cos a = \cdot 99989$ nearly.

Now $2\Delta DAB = xy \sin a = \cdot 01513xy$.

Also $2\Delta ADB = AC \times DB = 100 \times 6 = 600$;

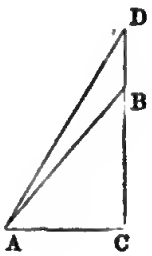
$$\therefore\ \cdot 015xy = 600,$$

$$xy = 4000$$

$$\cos a = \frac{x^2 + y^2 - 6^2}{2xy}, \quad \text{or } \cdot 9999 = \frac{x^2 + y^2 - 6^2}{2xy},$$

$$\cdot 9999 = \frac{x^2 + y^2 - 36}{80000} \quad \therefore\ x^2 + y^2 = 80028$$

$$2xy = 80000$$



$$x^2 + 2xy + y^2 = 160028$$
$$x^2 - 2xy + y^2 = 28$$
$$x + y = 400\cdot 035$$
$$x - y = \quad 5\cdot 291$$

$$2x = 405\cdot 326$$
$$x = 202\cdot 663$$
$$2y = 394\cdot 744$$
$$y = 197\cdot 372,$$

$$BC = \sqrt{(y^2 - AC^2)}$$
$$= \sqrt{(197\cdot 372^2 - 100^2)}$$
$$= \sqrt{28955\cdot 76384}$$
$$= 170\cdot 23188 \text{ feet.}$$

(21) From the top of a mountain 3 miles high, the visible horizon appeared depressed $2^\circ\ 13'\ 27''$; required the diameter of the earth, and the distance (d) of the horizon.

Referring to the figure and formulæ at page 64, we have

$$r = a \cot A \cot \tfrac{1}{2}A,$$

$$\frac{r}{d} = \cot A, \quad r = d \cot A;$$

$$\therefore\ a \cot A \cot \tfrac{1}{2}A = d \cot A$$

$$d = a \cot \tfrac{1}{2}A.$$

Here $a = 3$ miles, and $A = 2^\circ\ 13'\ 27''$;

$$\therefore\ d = 3 \cot 1^\circ\ 6'\ 43\tfrac{1}{2}''$$

$$\log 3 = 0{\cdot}47712$$

$$\log \cot 1^\circ\ 6'\ 43\tfrac{1}{2}'' = 11{\cdot}71194$$

$$12{\cdot}18906$$

$$10$$

$$2{\cdot}18906 = \log \text{ of } 154{\cdot}54;$$

$$\therefore\ d = 154{\cdot}54,$$

$$r = a \cot A \cot \tfrac{1}{2}A$$

$$= a \cot 2^\circ\ 13'\ 27'' \cot 1^\circ\ 6'\ 43\tfrac{1}{2}'',$$

$$\log r = \log a + \log \cot 2^\circ\ 13'\ 27'' + \log 1^\circ\ 6'\ 43\tfrac{1}{2}'' - 20$$

$$\log 3 = \cdot47712$$

$$\log \cot 2^\circ\ 13'\ 27'' = 11{\cdot}41075$$

$$\log \cot 1^\circ\ 6'\ 43\tfrac{1}{2}'' = 11{\cdot}71194$$

$$23{\cdot}59981$$

$$20$$

$$3{\cdot}59981 = \log 3979$$

$$\therefore\ r = 3979;\ \therefore\ 2r = 7958.$$

(22) A person, at a known distance from two towers, observes that their apparent altitudes are the same; he then walks a given distance towards them, till the angle of elevation of one is double that of the other; find the heights of the towers.

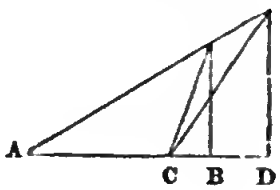
Let x and y = the heights,

$AB = a$, and α, β, the angles,

$AD = b$,

$CB = c$,

$CD = d$,



$$\tan \alpha = \frac{x}{a} = \frac{y}{b},\quad \tan \beta = \frac{y}{d},\quad \tan 2\beta = \frac{x}{c};$$

$$\text{but } \tan 2\beta = \frac{2 \tan \beta}{1 - \tan^2 \beta} = \frac{2\frac{y}{d}}{1 - \frac{y^2}{d^2}} = \frac{2dy}{d^2 - y^2} = \frac{x}{c};$$

$$\text{and } \because x = \frac{ay}{b},$$

$$\frac{2dy}{d^2 - y^2} = \frac{\frac{ay}{b}}{c} = \frac{ay}{bc}; \qquad \therefore \frac{2d}{d^2 - y^2} = \frac{a}{bc},$$

$$(d^2 - y^2)\, a = 2dcb; \qquad \therefore d^2 - y^2 = \frac{2dcb}{a},$$

$$y^2 = d^2 - \frac{2dcb}{a},$$

$$y = \sqrt{\left(d^2 - \frac{2dcb}{a}\right)},$$

$$x = \frac{ay}{b} = \frac{a}{b}\sqrt{\left(d^2 - \frac{2dcb}{a}\right)},$$

$$x = \frac{ad}{b}\sqrt{\left(1 - \frac{2cb}{ad}\right)}.$$

(23) In the arc AB of a circle, centre O, AC is taken = sin AB; then will sector COB = segment ACB.

The sector COB + sector COA

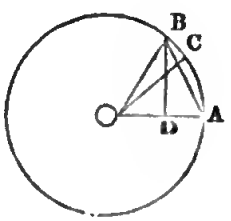
= ΔBOA + segment BCA,

or sector $COB + AC\frac{r}{2}$

= segment $ACB + BD\frac{r}{2}$,

but BD = arc AC;

$\therefore$ sector COB = segment ACB.



PRACTICAL REMARKS ON THE SOLUTION OF TRIANGLES.

23. In working out the various problems in the solution of triangles by the tables, when one or more logarithms have to be subtracted, it is better to write down their complements, or what each of them wants of 10, instead of the logarithms themselves, and then add them altogether, subtracting as many tens from the index of the sum as there are logarithms to be subtracted.

Thus, if the logarithm to be subtracted be 3·6947362, it is the same thing as to add its complement 6·3052638, and then to subtract 10 from the sum; the easiest way to perform the subtraction is to begin at the left hand, and subtract each figure from 9 except the last figure on the right, which must be subtracted from 10.

If the index of the logarithm be greater than 10, write down what it wants of 19, and take the rest of the figures from 9 as before; observing, that after the addition is made 20 must be taken from the index. Thus, the complement of the logarithm of 13·6874563 is 6·3125437. If the logarithm of a decimal which has a negative index is to be subtracted, add the index, considered as positive, to 9, and then find the complement of the rest of the figures as usual, and subtract 10 from the index of the sum. Thus, the complement of the logarithm of $\bar{2}$·3674586 is 11·6325414.

24. Before proceeding to the next examples we shall shew how the trigonometrical tables are constructed. (See Lefebure de Fourcy's Trigonométrie, p. 35.)

Let us at first find the sin 10″; we have before stated that if the diameter of a circle be unity the circumference = 3·141592653589793 = π, and when the radius is unity the semicircumference = π, and since there are 64800 seconds in 180°, we shall have in parts of the radius

$$\text{arc } 10'' = \frac{\pi}{64800} = \cdot 000048481368110.$$

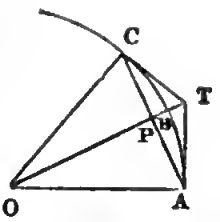
Now in very small arcs the sine and the arc are very nearly equal, hence the above may be considered as a good approximate value of the sin 10″.

25. We proceed now to shew that in the first quadrant, the arc is greater than the sine and less than the tangent.

Let AP be the sine of the arc AB, and AT the tangent, and suppose this figure to turn round OP till the point A falls on C, we shall have the arc $AC >$ the chord AC, and, consequently, the arc $AB > AP$; hence, the arc is greater than the sine. We have, also, the arc $AC < (AT + CT)$, and therefore $AB < AT$.

Hence it follows that if $\frac{\tan a}{\sin a}$ differs little from 1, the ratio $\frac{a}{\sin a}$ differs still less.

Also, as the arc decreases the ratio of that arc to its sine becomes as near to unity as we please; that is, this ratio has unity for its limit.



And since $\tan a = \frac{\sin a}{\cos a}$; $\therefore \frac{\tan a}{\sin a} = \frac{1}{\cos a}$.

Now, diminishing the arc (which is supposed less than 90^0) the cosine increases and approaches unity as nearly as we please, that is, the ratio $\frac{1}{\cos a}$ or its equal $\frac{\tan a}{\sin a}$ continually diminishes, and has unity for its limit.

The arc being greater than the sine and less than the tangent, the ratio $\frac{a}{\sin a}$ can never be less than 1 nor greater than $\frac{\tan a}{\sin a}$, and since this last ratio can approach unity as nearly as we please, the former one will also approach unity as nearly as we please, and it is on this account that we take the arc of $10''$ for the sin $10''$.

26. We must now proceed to determine the degree of approximation

$$\sin a = 2 \sin \tfrac{1}{2}a \cos \tfrac{1}{2}a, \text{ and } \tan \tfrac{1}{2}a > \tfrac{1}{2};$$

$$\text{or } \frac{\sin \frac{1}{2}a}{\cos \frac{1}{2}a} > \tfrac{1}{2}a; \therefore \sin \tfrac{1}{2}a > \tfrac{1}{2}a \cos \tfrac{1}{2}a;$$

$$\text{or } 2 \sin \tfrac{1}{2}a > a \cos \tfrac{1}{2}a;$$

$$\text{and } 2 \sin \tfrac{1}{2}a \cos \tfrac{1}{2}a > a \cos^2 \tfrac{1}{2}a, \text{ or } \sin a > a \cos^2 \tfrac{1}{2}a;$$

$$\text{but } \cos^2 \tfrac{1}{2}a = 1 - \sin^2 \tfrac{1}{2}a;$$

$$\therefore \sin a > a \left(1 - \sin^2 \tfrac{1}{2}a\right)$$

$$\sin a > a \left\{1 - \left(\tfrac{1}{2}a\right)^2\right\} > a - \frac{a^3}{4}.$$

Applying this result to the arc of 10″, taking the nearest value to five places of decimals, we shall have the arc of

$$10'' < \cdot 00005, \text{ then } \frac{1}{4}(\text{arc } 10'')^3 < \cdot 000000000000032;$$

hence we have

$$\sin 10'' > \{\cdot 000048481368110 - \cdot 000000000000032\};$$

$$\text{or } \sin 10'' > \cdot 000048481368078,$$

we see that this sine only begins to differ from the arc of 10″ at the 13th place of decimals, and taking the nearest decimal it is the same; hence it follows that we may take

$$\sin 10'' = \cdot 0000484813681,$$

and we may be assured that the error will be less than unity at the 13th place of decimals,

$$\text{and since } \cos 10'' = \sqrt{1 - \sin^2 10''}, \text{ we have}$$

$$\cos 10'' = \cdot 9999999988248.$$

27. We can now successively find the sines and cosines of 20″, 30″,... and so on up to 45°, by means of the formulæ (1) and (4), given at page 28,

$$\sin(a+b) = 2\cos b \sin a - \sin(a-b),$$

$$\cos(a+b) = 2\cos b \cos a - \cos(a-b);$$

we can consider the arcs $a-b$, a, $a+b$, as three consecutive terms of an arithmetical progression whose common difference is b, if we put t, t', t'' for these three terms, we have

$$\sin t'' = 2\cos b \sin t' - \sin t,$$

$$\cos t'' = 2\cos b \cos t' - \cos t;$$

consequently, to obtain the sine and cosine for every 10″ we must make $b = 10''$, and putting α and β for the known values of the sine and cosine of 10″, we have

$\sin 0 = 0$,	$\cos 0 = 1$,
$\sin 10'' = \alpha$,	$\cos 10'' = \beta$,
$\sin 20'' = 2\beta \sin 10''$,	$\cos 20'' = 2\beta \cos 10'' - 1$,
$\sin 30'' = 2\beta \sin 20'' - \sin 10''$,	$\cos 30'' = 2\beta \cos 20'' - \cos 10''$,
$\sin 40'' = 2\beta \sin 30'' - \sin 20''$,	$\cos 40'' = 2\beta \cos 30'' - \cos 20''$,
&c. &c.	&c. &c.

But as β differs very little from unity, 2β is nearly $= 2$, and the calculation can be still more abridged; put k for the difference $2 - 2\beta$, we shall have $k = \cdot 0000000023504$, and $2\beta = 2 - k$, consequently the value of $\sin t''$ becomes

$$\sin t'' = 2 \sin t' - k \sin t' - \sin t;$$

$$\therefore \sin t'' - \sin t' = (\sin t' - \sin t) - k \sin t'.$$

In such a long series of calculation the errors multiply considerably, and we therefore cannot be sure of the decimals being correct to the end, but, by using the formulæ given in Art. 15, we may verify the results, for the number of decimals common to both processes may be relied on as exact.

We may remark that when the radius is unity the sines and cosines will be fractional, and, consequently, their logarithms will be negative, in order to render them positive, the radius is taken $= 10^{10}$.

EXAMPLES.

In the right-angled triangle ACB, given the hypothenuse $AB = 1785{\cdot}395$ yards and the angle $ABC = 59^\circ\, 37'\, 42''$, to find the other two sides.

To find AC or b.	To find BC.
$b = c \sin B$,	$a = c \cos B$,
or $\log b = \log c + \log \sin B - 10$	$\log a = \log c + \log \cos B - 10$
$\log 1785{\cdot}395 \ \ldots\ 3{\cdot}2517343$	$\log 1785{\cdot}395 \ \ldots\ldots 3{\cdot}2517343$
$\log 59^\circ\, 37'\, 42'' \ldots\ 9{\cdot}9358919$	$\log \cos 59^\circ\, 37'\, 42'' \ldots 9{\cdot}7038132$
$\log b \qquad 3{\cdot}1876262$	$\log a \qquad 2{\cdot}9555475$
$\therefore\ b = 1540{\cdot}0374.$	$\therefore\ a = 902.708.$

In the triangle ABC (see fig. p. 52), given $a = 9459{\cdot}31$ yards, $b = 8032{\cdot}29$ yards, $c = 8242{\cdot}58$ yards, find A;

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(S-b)\,(S-c)}{bc}},$$

$2S = 25734{\cdot}18$, $S = 12867{\cdot}09$, $S - b = 4834{\cdot}8$, $S - c = 4624{\cdot}51$,

$$\log \sin \tfrac{1}{2} A = 10 + \frac{1}{2}\{\log (S-b) + \log (S-c) - \log b - \log c\},$$

$$2 \log \sin \tfrac{1}{2} A = 20 + \log (S-b) + \log (S-c) - \log b - \log c,$$

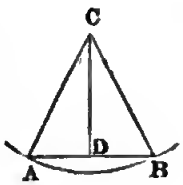
$$\begin{aligned}
\log(S-b) &= 3{\cdot}6843785\\
\log(S-c) &= 3{\cdot}6650657\\
\text{Comp. } \log b &= 6{\cdot}0951606\\
\text{Comp. } \log c &= 6{\cdot}0839368\\
\hline
2\log\sin\tfrac{1}{2}A &= 19{\cdot}5285416,\\
\log\sin\tfrac{1}{2}A &= 9{\cdot}7642708,\\
\tfrac{1}{2}A &= 35^\circ 31' 47'',\\
A &= 71^\circ 3' 34''.
\end{aligned}$$

CHAPTER IV.

ON POLYGONS.

28. To find the radius of a circle described about a regular polygon of n sides.

Let $R = AC =$ radius, AB one of the sides of the polygon $= a$, ACB the angle subtending one of the sides of the polygon, then n times the angle ACB will be the sum of all the angles, or 360°;



$$\therefore \angle ACB = \frac{360^\circ}{n}; \quad \therefore \angle ACD = \frac{180^\circ}{n},$$

since CD bisects the angle ACB,

$$R\sin ACD = AD, \text{ or } 2R\sin ACD = AB;$$

$$\therefore R = \frac{AB}{2\sin ACD} = \frac{AB}{2\sin\frac{180^\circ}{n}} = \frac{1}{2}AB\operatorname{cosec}\frac{180^\circ}{n}$$

$$= \frac{1}{2}a\operatorname{cosec}\frac{180^\circ}{n}.$$

29. If now we suppose AB to be the side of a regular polygon described about a circle, then CD will be the radius of that circle, which call r,

$$\text{then } \frac{CD}{AD} = \cot ACD = \cot\frac{180^\circ}{n};$$

$$\therefore\ CD = r = AD\cot\frac{180^0}{n} = \frac{1}{2}AB\cot\frac{180^0}{n} = \frac{1}{2}a\cot\frac{180^0}{n};$$

$$\therefore\ R + r = \frac{1}{2}a\left(\cot\frac{180^0}{n} + \operatorname{cosec}\frac{180^0}{n}\right)$$

$$= \frac{1}{2}a\left\{\frac{\cos\frac{180^0}{n}}{\sin\frac{180^0}{n}} + \frac{1}{\sin\frac{180^0}{n}}\right\}$$

$$= \frac{1}{2}a\left\{\frac{1+\cos\frac{180^0}{n}}{\sin\frac{180^0}{n}}\right\} = \frac{\frac{a}{2}\left(2\cos^2\frac{90^0}{n}\right)}{2\sin\frac{90^0}{n}\cos\frac{90^0}{n}} = \frac{a}{2}\cot\frac{90^0}{n},$$

$$R.r = \frac{1}{4}a^2\operatorname{cosec}\frac{180^0}{n}\cot\frac{180^0}{n} = \frac{1}{4}a^2\frac{\operatorname{cosec}^2\frac{180^0}{n}}{\sec\frac{180^0}{n}},$$

$$\frac{R}{r} = \frac{\frac{1}{2}a\operatorname{cosec}\frac{180^0}{n}}{\frac{1}{2}a\cot\frac{180^0}{n}} = \frac{\operatorname{cosec}\frac{180^0}{n}}{\cos\frac{180^0}{n}\operatorname{cosec}\frac{180^0}{n}} = \frac{1}{\cos\frac{180^0}{n}}.$$

30. To find the area of a regular polygon of n sides inscribed in a circle.

The polygon is composed of as many triangles as there are sides of the figure:

area of the triangle $ACB = AD.CD = r\sin\frac{180^0}{n}\,r\cos\frac{180^0}{n}$

$$= r^2\sin\frac{180^0}{n}\cos\frac{180^0}{n} = \frac{1}{2}r^2\sin\frac{360^0}{n}.$$

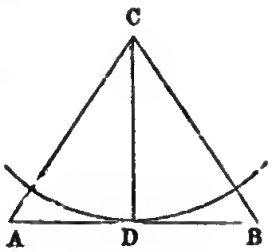
Hence the area of the inscribed polygon = n times the triangle $ACB = \frac{n}{2}r^2\sin\frac{360^0}{n}$.

Or thus, the area of the triangle $ACB = \frac{AC.CB\sin ACB}{2}$

$$= \frac{r^2\sin\frac{360^0}{n}}{2};$$

$$\therefore \text{ area of polygon} = \frac{n}{2} r^2 \sin \frac{360^0}{n}.$$

31. For the circumscribing polygon.



Let $CD=r$, $\frac{AD}{CD} = \tan ACD = \tan \frac{180^0}{n}$;

$$\therefore AD = CD \tan \frac{180^0}{n} = r \tan \frac{180^0}{n},$$

$$\text{area of } \triangle ACB = AD.CD = r^2 \tan \frac{180^0}{n},$$

$$\text{area of polygon} = n \text{ times the triangle } ACB = nr^2 \tan \frac{180^0}{n},$$

$$\frac{\text{area of the inscribed polygon}}{\text{area of the circumscribed polygon}} = \frac{\frac{n}{2} r^2 \sin \frac{360^0}{n}}{nr^2 \tan \frac{180^0}{n}} = \frac{\frac{1}{2} \sin \frac{360^0}{n}}{\tan \frac{180^0}{n}}$$

$$= \frac{\frac{1}{2}.2 \sin \frac{180^0}{n} \cos \frac{180^0}{n}}{\frac{\sin \frac{180^0}{n}}{\cos \frac{180^0}{n}}} = \cos^2 \frac{180^0}{n}.$$

The area of a circle whose radius is $r = \pi r^2$.

From the above the area of a polygon inscribed in a circle, is

$$\frac{n}{2} r^2 \sin \frac{360^0}{n} = \frac{n}{2} r^2 \sin \frac{2\pi}{n} = \pi r^2 \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}}.$$

As we increase the number of sides, the area of the polygon approaches nearer and nearer to that of the circle, and when n is infinitely great it becomes equal to it, for then

$\frac{2\pi}{n}$ is an infinitely small angle, therefore $\frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} = 1$;

$\therefore$ area of the circle $= \pi r^2$.

Examples on the two preceding Chapters.

(1) Given the perimeter of a right-angled triangle, and the radius of the inscribed circle; determine the sides,

$$S = \frac{a+b+c}{2}, \qquad \frac{ab}{2} = \Delta = Sr,$$

$$ab = 2Sr, \qquad a+b+c = 2S, \qquad a+b = 2S-c,$$

$$a+b = 2S-c = 2S-(S-r) = S+r,$$

$$a+b = S+r,$$

$$a^2+2ab+b^2 = S^2+2Sr+r^2,$$

$$4ab \quad = \quad 8Sr;$$

$$a^2-2ab+b^2 = S^2-6Sr+r^2,$$

$$a-b = \sqrt{(S^2-6Sr+r^2)}$$

$$a+b = 2S-c,$$

$$2a = 2S-c+\sqrt{(S^2-6Sr+r^2)},$$

$$a = \frac{2S-c+\sqrt{(S^2-6Sr+r^2)}}{2},$$

$$2b = 2S-c-\sqrt{(S^2-6Sr+r^2)},$$

$$b = \frac{2S-c-\sqrt{(S^2-6Sr+r^2)}}{2},$$

$$c = S-r$$

$$a^2+2ab+b^2 = 4S^2-4Sc+c^2$$

$$a^2 \qquad +b^2 = \qquad\qquad c^2$$

$$2ab = 4S^2-4Sc, \qquad \text{but } ab = 2Sr,$$

$$\therefore\ 4Sr = 4S^2-4Sc$$

$$4Sc = 4S^2-4Sr$$

$$4c = 4S \ -4r$$

$$c = \ S \ -r.$$

(2) Given the perimeter of the same triangle, and the radius of the circumscribed circle; find the sides,

$$\frac{a+b+c}{2} = S, \ \frac{ab}{2} = \Delta = \frac{abc}{4R},$$

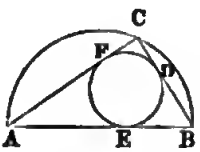
$$a+b+c = 2S \qquad 2R = c,$$

$$a+b = 2S-2R,$$

$$a^2 + 2ab + b^2 = 4S^2 - 8SR + 4R^2,$$
$$a^2 \quad + b^2 = \quad 4R^2;$$
$$\therefore\ 2ab \quad = 4S^2 - 8SR;$$
$$\therefore\ a^2 - 2ab + b^2 = 4R^2 + 8SR - 4S^2;$$
$$\therefore\ a - b = \sqrt{(4R^2 + 8SR - 4S^2)},$$
$$a + b = 2S - 2R,$$
$$2a = 2S - 2R + \sqrt{(4R^2 + 8SR - 4S^2)},$$
$$a = \frac{2S - 2R + \sqrt{(4R^2 + 8SR - 4S^2)}}{2},$$
$$2b = 2S - 2R - \sqrt{(4R^2 + 8SR - 4S^2)},$$
$$b = \frac{2S - 2R - \sqrt{(4R^2 + 8SR - 4S^2)}}{2}.$$

(3) If (D) and (d) be the diameters of the two circles, and (a), (b) the sides which include the right angle; shew that $D + d = a + b$.

$AF + FC + CD + DB = b + a$,
$AF = AE$ and $BE = BD$,
$FC = r$ and $DC = r$;
$\therefore\ AE + BE + 2r = a + b$,
but $AE + EB = D$.



Hence $a + b = D + 2r = D + d$.

(4) The sides of a triangle are as 3, 5, 6; shew that

$$R : r :: 45 : 16,$$

$$Rr = \frac{abc}{2(a+b+c)} = \frac{3.5.6}{2(3+5+6)} = \frac{3.5.3}{3+5+6} = \frac{45}{14},$$

$$r = \sqrt{\frac{(S-a)(S-b)(S-c)}{S}} = \sqrt{\frac{(7-3)(7-5)(7-6)}{7}} = \sqrt{\frac{8}{7}},$$

$$R = \frac{45}{14} \div r = \frac{45}{14} \div \sqrt{\frac{8}{7}} = \frac{45}{14} \times \sqrt{\frac{7}{8}};$$

$$R : r :: \frac{45}{14} \times \sqrt{\frac{7}{8}} : \sqrt{\frac{8}{7}},$$

$$45 \times 7 : 14 \times 8;$$

$$R : r :: 45 : 16.$$

(5) If the side of a pentagon inscribed in a circle be 1, the radius $= \frac{\sqrt{(5+\sqrt{5})}}{\sqrt{10}}$.

Here $n = 5$, $a = 1$,

$$\text{perimeter} = 5 = 2nr \sin \frac{\pi}{n}$$

$$= 2 \times 5r \sin \frac{180^\circ}{5},$$

$$\therefore\ 1 = 2r \sin 36^\circ$$

$$= 2r \frac{\sqrt{(5-\sqrt{5})}}{2\sqrt{2}}$$

$$= r \frac{\sqrt{(5-\sqrt{5})}}{\sqrt{2}},$$

$$\therefore\ r = \frac{1}{\frac{\sqrt{(5-\sqrt{5})}}{\sqrt{2}}}$$

$$= \frac{\sqrt{2}}{\sqrt{(5-\sqrt{5})}}$$

$$= \frac{\sqrt{2}\sqrt{(5+\sqrt{5})}}{\sqrt{(25-5)}},$$

$$= \frac{\sqrt{2}\sqrt{(5+\sqrt{5})}}{\sqrt{20}},$$

$$= \frac{\sqrt{(5+\sqrt{5})}}{\sqrt{10}}.$$

(6) Find the actual value of a side of a twenty-four sided regular figure inscribed in a given circle.

Here $n = 24$. Put $A = \frac{15^\circ}{2}$, $2A = 15^\circ$,

$$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

$$\text{Perimeter} = 2nr \sin\frac{\pi}{n}$$

$$= 2 \times 24r \sin\frac{180^\circ}{24}$$

$$= 2 \times 24r \sin\frac{15^\circ}{2},$$

but $\sin A = \frac{1}{2}\{\sqrt{(1+\sin 2A)} - \sqrt{(1-\sin 2A)}\}$

$$= \frac{1}{2}\left\{\sqrt{\left(1+\frac{\sqrt{3}-1}{2\sqrt{2}}\right)} - \sqrt{\left(1-\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}\right\};$$

$$\therefore \text{perimeter} = 2 \times 24r \frac{1}{2}\left\{\sqrt{\left(1+\frac{\sqrt{3}-1}{2\sqrt{2}}\right)} - \sqrt{\left(1-\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}\right\}$$

$$= 2 \times 12r\left\{\sqrt{\left(1+\frac{\sqrt{3}-1}{2\sqrt{2}}\right)} - \sqrt{\left(1-\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}\right\}$$

$$= 24r\left\{\sqrt{\left(1+\frac{\sqrt{3}-1}{2\sqrt{2}}\right)} - \sqrt{\left(1-\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}\right\},$$

but $a = \frac{\text{perimeter}}{24}$;

$$\therefore a = 24r\frac{\sqrt{\left(1+\frac{\sqrt{3}-1}{2\sqrt{2}}\right)} - \sqrt{\left(1-\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}}{24},$$

$$= r\left\{\sqrt{\left(1+\frac{\sqrt{3}-1}{2\sqrt{2}}\right)} - \sqrt{\left(1-\frac{\sqrt{3}-1}{2\sqrt{2}}\right)}\right\}.$$

(7) If r be the radius of the inscribed circle,

$$\text{area} = r^2(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C).$$

By Euc. Book IV. Prop. 4,

$$\angle OBA = \tfrac{1}{2}B, \ \angle OCA = \tfrac{1}{2}C.$$

Now $Ba = Oa \cot\frac{1}{2}B$, $Ca = Oa \cot\frac{1}{2}C$;

$$\therefore a = Oa \cot\tfrac{1}{2}B + Oa \cot\tfrac{1}{2}C,$$

$$a = r(\cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C),$$

$$b = r(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}C),$$

$$c = r(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B),$$

$$a+b+c = r(\cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C) + r(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}C) + r(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B),$$

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$$S = \frac{r(\cot\frac{1}{2}B + \cot\frac{1}{2}C) + r(\cot\frac{1}{2}A + \cot\frac{1}{2}C) + r(\cot\frac{1}{2}A + \cot\frac{1}{2}B)}{2}$$

$$= r\,\frac{(2\cot\frac{1}{2}A + 2\cot\frac{1}{2}B + 2\cot\frac{1}{2}C)}{2}$$

$$= 2r\,\frac{(\cot\frac{1}{2}A + \cot\frac{1}{2}B + \cot\frac{1}{2}C)}{2},$$

$$S = r\,(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C).$$

But area $= S.r$,

area $= rr\,(\cot\frac{1}{2}A + \cot\frac{1}{2}B + \cot\frac{1}{2}C)$;

$\therefore$ area $= r^2\,(\cot\frac{1}{2}A + \cot\frac{1}{2}B + \cot\frac{1}{2}C)$.

(8) The perimeters of an equilateral triangle, a square, and a hexagon, each including the same area, are as $\sqrt[4]{27}$, $\sqrt[4]{16}$, $\sqrt[4]{12}$.

Let each side of the $\triangle ABC$ at p. 82 $= a$, $3a =$ perimeter,

$$CD = \sqrt{a^2 - \left(\frac{a}{2}\right)^2} = \sqrt{\frac{3a^2}{4}},\ \text{and}\ \frac{CD.a}{2} = \text{area}\ \triangle;$$

$\therefore$ side of equal square $= \sqrt{\dfrac{CD.a}{2}}$

$$= \sqrt{CD}.\sqrt{\frac{a}{2}} = \sqrt{\frac{3a^2}{4}}.\sqrt[4]{\frac{a^2}{4}} = \sqrt[4]{\frac{3a^4}{16}},$$

$$4.\sqrt[4]{\frac{3a^4}{16}} = \text{perimeter of square.}$$

$$\text{Area of}\ \triangle = CD.\frac{a}{2} = \sqrt{\frac{3a^2}{4}}.\sqrt{\frac{a^2}{4}} = \sqrt{\frac{3a^4}{16}}$$

= area of square = area of hexagon. Let $x =$ side of hexagon.

$$\text{Area of hexagon} = \frac{nx^2}{4}.\cot\frac{\pi}{n} = \frac{6x^2}{4}.\cot\frac{180^\circ}{6} = \frac{6x^2}{4}\cot 30^\circ$$

$$= \frac{3}{2}x^2.\sqrt{3} = \sqrt{\frac{3a^4}{16}},\ \ \frac{27x^4}{4} = \frac{3a^4}{16},\ \ 36x^4 = a^4,$$

$$6x^2 = a^2,\ \ x\sqrt{6} = a;\ \ \therefore\ x = \frac{a}{\sqrt{6}} = \text{side of hexagon:}$$

$$6\frac{a}{\sqrt{2}} = a\sqrt{6} = a\sqrt{3}.\sqrt{2} = \text{perimeter of hexagon:}$$

$$3a\ :\ 2a\sqrt[4]{3}\ :\ a\sqrt{2}.\sqrt{3} = \sqrt[4]{27}\ :\ \sqrt[4]{16}\ :\ \sqrt[4]{12}.$$

(9) If O be the centre of the circle inscribed in the triangle ABC,

$$OA^2 + OB^2 + OC^2 = ab + ac + bc - 12Rr.$$

See fig. page 59.

$$\left.\begin{aligned} AO^2 &= r^2 + AE^2 = r^2 + r^2 \cot^2 \tfrac{1}{2}A, \\ OB^2 &= r^2 + BF^2 = r^2 + r^2 \cot^2 \tfrac{1}{2}B, \\ OC^2 &= r^2 + EC^2 = r^2 + r^2 \cot^2 \tfrac{1}{2}C, \end{aligned}\right\}$$

$$AO^2 + OB^2 + OC^2 = 3r^2 + r^2\left(\cot^2\frac{A}{2} + \cot^2\frac{B}{2} + \cot^2\frac{C}{2}\right),$$

by Art. 21, and p. 55,

$$= 3\frac{(s-a)(s-b)(s-c)}{s} + \frac{(s-a)(s-b)(s-c)}{s}$$

$$\left\{\frac{s(s-a)}{(s-b)(s-c)} + \frac{s(s-b)}{(s-a)(s-c)} + \frac{s(s-c)}{(s-b)(s-a)}\right\}$$

$$= 3\frac{(s-a)(s-b)(s-c)}{s} + (s-a)^2 + (s-b)^2 + (s-c)^2$$

$$= 3s^2 - 3sa - 3sb + 3ab - 3sc + 3ac + 3bc - \frac{3abc}{s}$$

$$+ s^2 - 2sa + a^2 + s^2 - 2sb + b^2 + s^2 - 2sc + c^2$$

$$= 6s^2 - 5s(a+b+c) + a^2 + b^2 + c^2 + 2ab + 2ac$$

$$+ 2bc + ab + ac + bc - \frac{3abc}{s}$$

$$= 6s^2 - 5s \cdot 2s + (a+b+c)^2 + ab + ac + bc - 12Rr$$

$$= 6s^2 - 10s^2 + 4s^2 + ab + ac + bc - 12Rr$$

$$= ab + ac + bc - 12Rr;$$

$$\therefore\ AO^2 + OB^2 + OC^2 = ab + ac + bc - 12Rr.$$

(10) The area of a regular hexagon inscribed in a circle is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

$$\text{Area of hexagon} = \frac{nr^2}{2}\sin\frac{2\pi}{n}$$

$$= \frac{6r^2}{2}\sin\frac{360^\circ}{6}$$

$$= 3r^2 \sin 60^\circ$$

$$= 3r^2\frac{\sqrt{3}}{2}.$$

$$\text{Area of inscribed triangle} = \frac{nr^2}{2}\sin\frac{2\pi}{n}$$
$$= \frac{3r^2}{2}\sin\frac{360^\circ}{3}$$
$$= \frac{3r^2}{2}\sin 120^\circ$$
$$= \frac{3r^2}{2}\frac{\sqrt{3}}{2}$$
$$= \frac{3r^2\sqrt{3}}{4}.$$

$$\text{Area of circumscribed } \Delta = nr^2\tan\frac{\pi}{2}$$
$$= 3r^2\tan 60^\circ$$
$$= 3r^2\sqrt{3};$$
$$\therefore \text{ as } \frac{3r^2\sqrt{3}}{4} : \frac{3r^2\sqrt{3}}{2} :: \frac{3r^2\sqrt{3}}{2} : 3r^2\sqrt{3}.$$

(11) The square of the side of a pentagon inscribed in a circle is equal to the sum of the squares of the sides of a regular hexagon and decagon inscribed in the same circle.

$$\text{Pentagon, perimeter} = n \times AB = 2nr\sin\frac{\pi}{n};$$

$$AB = 2r\sin\frac{180^\circ}{5} = 2r\sin 36^\circ$$
$$= 2r\frac{\sqrt{(5-\sqrt{5})}}{2\sqrt{2}} = r\frac{\sqrt{(5-\sqrt{5})}}{\sqrt{2}},$$
$$AB^2 = \frac{(5-\sqrt{5})r^2}{2}.$$

$$\text{Hexagon, perimeter} = n\,AB_1 = 2nr\sin\frac{\pi}{n},$$

$$AB_1 = 2r\sin\frac{180^\circ}{6} = 2r\sin 30^\circ$$
$$= 2r\frac{1}{2} = r; \quad \therefore AB_1^2 = r^2.$$

Decagon, perimeter $= n\,AB_{,,} = 2nr \sin \frac{180^\circ}{n}$,

$$AB_{,,} = 2r \sin \frac{180^\circ}{10} = 2r \sin 18^\circ$$

$$= 2r \frac{\sqrt{5}-1}{4} = r \frac{\sqrt{5}-1}{2};$$

$$\therefore\ AB_{,,}^2 = \frac{(3-\sqrt{5})\,r^2}{2},$$

$$AB_{,}^2 + AB_{,,}^2 = \frac{(3-\sqrt{5})\,r^2}{2} + r^2 = \frac{(3-\sqrt{5}+2)\,r^2}{2}$$

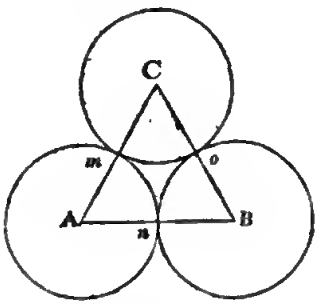
$$= \frac{(5-\sqrt{5})\,r^2}{2} = AB^2.$$

(12) Three equal circles touch each other; shew that the space between them is nearly equal to the square described upon a fifth part of the diameter; find the area when the circles are unequal, and the radii as the numbers 1, 2, 3.

Let rad. $= \frac{1}{2}$ diam. $= 1$, $\pi = 3{\cdot}1416$,

$$\text{arc } mn = \frac{2\,(3{\cdot}1416)}{6} = {\cdot}5236 \times 2$$

$$= 1{\cdot}0472.$$



Sector $Amn = \frac{1}{2} r \times \text{arc } mn = {\cdot}5236$;

$\therefore$ the area of the three sectors $= 1{\cdot}5708$,

$Cn^2 = AC^2 - An^2$; $\quad\therefore\ Cn = \sqrt{(AC^2 - An^2)}$

$= \sqrt{(AC^2 - 1)} = \sqrt{(4-1)} = \sqrt{3}$;

$$\Delta ABC = \frac{AB \times Cn}{2} = 1{\cdot}73205.$$

Now the space $mno = \Delta ABC - 3$ sector Amn

$= 1{\cdot}73205 - 1{\cdot}5708 = {\cdot}161.$

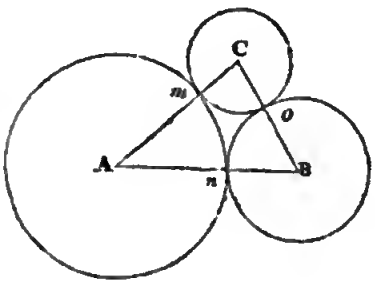
Now the square described on $\frac{1}{5}$ of the diam. $= \frac{4}{25} = {\cdot}16.$

Hence space $mno = \left(\frac{\text{diam.}}{5}\right)^2$ nearly.

(13) Let rad. $Co=1$, $Bo=2$, $Am=3$, then the diameters are 2, 4, 6; and the sides of the ΔABC are

$$BC=3,\ AC=4,\ AB=5,\ \therefore\ AB^2=AC^2+BC^2;$$

$$\therefore\ \angle C=90^\circ,\ \sin a=\frac{BC}{AB},$$



$$\log\sin A=\log BC-\log AB+10,$$

$$\log 3=\ \cdot47712$$

$$\log 5=\ \cdot69897$$

$$10$$

$$\log\sin 36^\circ 52'=\ 9\cdot77815\ ;$$

$$\therefore\ A=36^\circ\cdot8666,$$

$$B=53^\circ\cdot1333,$$

$$360^\circ\ :\ 36^\circ\cdot866\ ::\ 3\cdot1416\times6\ :\ 1\cdot9305=\text{arc } mn,$$

$$\text{sector } Amn=\frac{1\cdot9305\times3}{2}=2\cdot8958,$$

$$360^\circ\ :\ 53^\circ\cdot133\ ::\ 3\cdot1416\times4\ :\ 1\cdot8546=\text{arc } no,$$

$$\text{sector } Bno=\frac{1\cdot8546\times2}{2}=1\cdot8546,$$

$$\frac{\text{circumference of }\odot moC}{4}=\frac{3\cdot1416\times2}{4}=1\cdot5708=\text{arc } mo,$$

$$\text{sector } Com=\frac{1\cdot5708\times1}{2}=\cdot7854,$$

$$2\cdot8958$$

$$1\cdot8546$$

$$\cdot7854$$

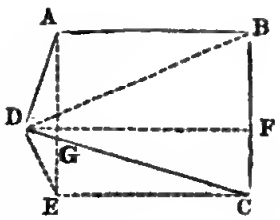
$$\text{sum of sectors}=5\cdot5358$$

$$\Delta ABC=\frac{AC\times BC}{2}=\frac{4\times3}{2}=6;$$

$\therefore$ space $nmo=\Delta ABC-$ sum of sectors $=\cdot4642$.

(14) Given $AB=a$, $BC=b$, $CD=c$, three sides of a quadrilateral, and $\angle ABC=\theta$, $\angle BCD=\phi$; shew that twice the area $=ab\sin\theta+cb\sin\phi-ac\,(\theta+\phi)$.

Let $ABCD$ be the quadrilateral.



From the point A draw $AE \parallel BC$, and from the point D draw $DF \parallel$ to AB or EC,

then $cb \sin\phi = 2\triangle BCD$,

and $ab \sin\theta =$ the figure $ABCE$

$= \square\, ABFG + GFCE$,

$= 2\triangle ABD + \square\, GFCE$;

$\therefore\ ab \sin\theta + bc \sin\phi = 2\triangle(ABD + BCD) + \square\, GFCE$

$= 2$ quadri. $+ \square\, GFCE$

$= 2$ quadri. $+ 2\triangle DEC$

$= 2$ quadri. $+ ac \sin(\phi + \theta)$;

$\therefore\ ab \sin\theta + bc \sin\phi - ac \sin(\phi + \theta) = 2$ quadrilateral.

(15) If R and r be the radii of the circumscribed and inscribed circles of a triangle:

$$r = 4R \sin\tfrac{1}{2}A \sin\tfrac{1}{2}B \sin\tfrac{1}{2}C.$$

See Ex. p. 86.

$$\tfrac{1}{2}(a+b+c) = r\{\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C\},$$

$$\text{area} = r^2\{\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C\},$$

$$\text{but } \tfrac{1}{2}A + \tfrac{1}{2}B + \tfrac{1}{2}C = 90^\circ;$$

$\therefore$ by Ex. 26, p. 45,

$$\text{area} = r^2\{\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C\}$$

$$= r^2 \cot\tfrac{1}{2}A \cot\tfrac{1}{2}B \cot\tfrac{1}{2}C.$$

And by Art. 22, Chap. III., and formulæ, p. 55,

$$\text{area} = 2R^2 \sin A \sin B \sin C$$

$$= 4R^2 \sin\tfrac{1}{2}A \cos\tfrac{1}{2}A\, 2\sin\tfrac{1}{2}B \cos\tfrac{1}{2}B\, 2 \sin\tfrac{1}{2}C \cos\tfrac{1}{2}C$$

$$= 16R^2 \sin\tfrac{1}{2}A \cos\tfrac{1}{2}A \sin\tfrac{1}{2}B \cos\tfrac{1}{2}B \sin\tfrac{1}{2}C \cos\tfrac{1}{2}C$$

$$= r^2 \cot\tfrac{1}{2}A \cot\tfrac{1}{2}B \cot\tfrac{1}{2}C$$

$$= r^2 \frac{\cos\frac{1}{2}A \cos\frac{1}{2}B \cos\frac{1}{2}C}{\sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C};$$

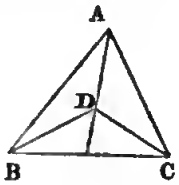
$$\therefore\ r^3 \frac{1}{\sin\frac{1}{2}A} \frac{1}{\sin\frac{1}{2}B} \frac{1}{\sin\frac{1}{2}C} = 16R^2 \sin\tfrac{1}{2}A \sin\tfrac{1}{2}B \sin\tfrac{1}{2}C,$$

$$r^3 \frac{1}{\sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C} = 16R^2 \sin\tfrac{1}{2}A \sin\tfrac{1}{2}B \sin\tfrac{1}{2}C;$$

$$\therefore\ r^2 = 16R^2 (\sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C)^2;$$

$$\therefore\ r = 4R \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C.$$

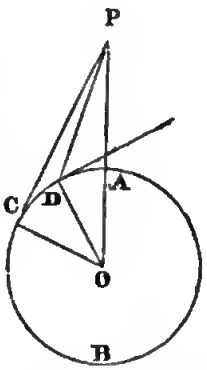
(16) In the triangle ABC take a point D, join DA, DB, DC; given AB, AC, the $\angle ABD$, $\angle ACD$, and $\angle BDC$, find BC.



$$AB = a, \qquad \angle ABD = \alpha,$$
$$AC = b, \qquad \angle ACD = \beta,$$
$$\angle BDC = \gamma,$$

$$\angle DBC + \angle DCB = \pi - \gamma,$$
$$\angle ABC + \angle ACB = (\pi - \gamma + \alpha + \beta),$$
$$\angle BAC = \pi - (\pi - \gamma + \alpha + \beta),$$
$$\angle BAC = \gamma - \alpha - \beta,$$
$$BC = \sqrt{(AB^2 + AC^2 - 2AB.AC \cos BAC)},$$
$$BC = \sqrt{\{a^2 + b^2 - 2ab \cos (\gamma - \alpha - \beta)\}}.$$

(17) $ABCD$ is the circumference of a circle, O its centre, from C draw a tangent to the circle meeting the radius OA produced, in P, join PD; then if $CP = a$, $DP = b$, θ = angle formed by DP and a tangent at D, prove that $r = \dfrac{a^2 - b^2}{2b} \operatorname{cosec} \theta$, r being the radius of the circle.



COP is a right-angled triangle;

$$\therefore\ OP^2 = a^2 + r^2,$$

and in $\triangle ODP$, the sides OD, DP, and $\angle ODP$ are known.

$$\therefore\ OP^2 = b^2 + r^2 - 2br \cos\left(\frac{\pi}{2} + \theta\right);$$

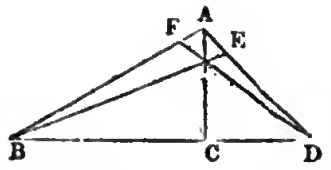
$$\therefore\ a^2 + r^2 = b^2 + r^2 - 2br \cos\left(\frac{\pi}{2} + \theta\right);$$

$$\therefore\ a^2 - b^2 = 2br \sin \theta;$$

$$\therefore\ r = \frac{a^2 - b^2}{2b \sin \theta} = \frac{a^2 - b^2}{2b} \operatorname{cosec} \theta.$$

(18) Given the three perpendiculars from the angles on the opposite sides; find the sides.

Let $AB = x$, $AD = y$, $BD = z$,
$AC = a$, $BE = b$, $DF = c$,
$2\Delta = cx = by = az$,
$2\Delta = 2\sqrt{\{S(S-x)(S-y)(S-z)\}}$;



$$\therefore\ cx = 2\sqrt{\{S(S-x)(S-y)(S-z)\}}$$

$$= 2\sqrt{\left(\frac{x+y+z}{2}\cdot\frac{y+z-x}{2}\cdot\frac{x+z-y}{2}\cdot\frac{x+y-z}{2}\right)}$$

$$= \frac{1}{2}\sqrt{\left\{\left(x+\frac{cx}{b}+\frac{cx}{a}\right)\left(\frac{cx}{b}+\frac{cx}{a}-x\right)\left(x+\frac{cx}{a}-\frac{cx}{b}\right)\left(x+\frac{cx}{b}-\frac{cx}{a}\right)\right\}}$$

$$= \frac{1}{2}x^2\sqrt{\left\{\left(1+\frac{c}{b}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{c}{a}-1\right)\left(1+\frac{c}{a}-\frac{c}{b}\right)\left(1+\frac{c}{b}-\frac{c}{a}\right)\right\}},$$

$$2c = x\sqrt{\left\{\left(1+\frac{c}{b}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{c}{a}-1\right)\left(1+\frac{c}{a}-\frac{c}{b}\right)\left(1+\frac{c}{b}-\frac{c}{a}\right)\right\}};$$

$$\therefore\ x = \frac{2c}{\sqrt{\left\{\left(1+\frac{c}{b}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{c}{a}-1\right)\left(1+\frac{c}{a}-\frac{c}{b}\right)\left(1+\frac{c}{b}-\frac{c}{a}\right)\right\}}}.$$

Similarly,

$$y = \frac{2b}{\sqrt{\left\{\left(1+\frac{b}{a}+\frac{b}{c}\right)\left(\frac{b}{a}+\frac{b}{c}-1\right)\left(1+\frac{b}{c}-\frac{b}{a}\right)\left(1+\frac{b}{a}-\frac{b}{c}\right)\right\}}},$$

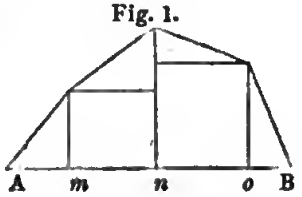
$$z = \frac{2a}{\sqrt{\left\{\left(1+\frac{a}{b}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{a}{c}-1\right)\left(1+\frac{a}{c}-\frac{a}{b}\right)\left(1+\frac{a}{b}-\frac{a}{c}\right)\right\}}}.$$

(19) If l = the base of a polygon, abc, &c., the sides beginning from the base, $\alpha\beta\gamma$, &c., the angles made by abc, &c., with the base, then

$$l = a\cos\alpha + b\cos\beta + c\cos\gamma + \&c.$$

Let $AB = l$, from the upper angles draw lines perpendicular and parallel to the base. Then (fig. 1),

Fig. 1.



$$Am = a\cos\alpha,\ mn = b\cos\beta,$$
$$no = c\cos\gamma,\ oB = d\cos\delta,$$

$$AB = Am + mn + no + oB = a\cos\alpha + b\cos\beta + c\cos\gamma + d\cos\delta;$$

$$\therefore\ l = a\cos\alpha + b\cos\beta + c\cos\gamma + \&c.$$

Fig. 2. $\frac{Am}{a} = \cos(\alpha - \pi) = -\cos\alpha$

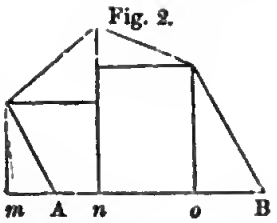


Fig. 2.

$-Am = a\cos\alpha$, $mn = Am + An = b\cos\beta$,

$no = c\cos\gamma$, $oB = d\cos\delta$,

$AB = -Am + Am + An + no + oB$

$= a\cos\alpha + b\cos\beta + c\cos\beta + \&c.$;

$\therefore\ l = a\cos\alpha + b\cos\beta + c\cos\gamma + \&c.$

32. The two following rules may be found useful to practical men; the first is when two sides and the included angle are given, and the second when the three sides are given.

By Art. 19, $$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}.$$

Hence the following rule:

As the sum of any two sides of a plane triangle is to their difference, so is the tangent of half the sum of their opposite angles to the tangent of half their difference.

By Art. 16, $$\cos A = \frac{b^2 + c^2 - a^2}{2bc};$$

$$\therefore\ a^2 - b^2 = c^2 - 2bc\cos A$$

$$= C(c^2 - 2b\cos A),$$

$$(a+b)(a-b) = C(c - 2b\cos A) = c(AD \sim DB),$$

$$c : a+b :: a-b : AD \sim DB.$$

From this we have the following rule:

Take the longest side for the base; then as the base is to the sum of the other two sides, so is their difference to the difference of the segments of the base.

Half the sum of the segments added to half their difference gives the greater segment, and half their difference subtracted from half their sum gives the least.

The following logarithmic forms for right-angled triangles may also be useful, although they are only the definitions at page 8 put into logarithms.

Referring to the figure, page 52, considering C to be a right angle, we have by the definitions,

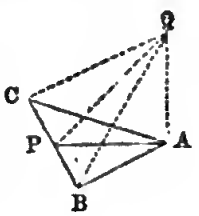
$$\sin A = \frac{BC}{AB} = \frac{a}{c}, \quad \cos A = \frac{AC}{AB} = \frac{b}{c}, \quad \tan A = \frac{BC}{AC} = \frac{a}{b}:$$

in logarithms,

$$\log a = \log c + \log \sin A - 10 = \log b + \log \tan A - 10,$$
$$\log b = \log c + \log \cos A - 10 = \log a - \log \tan A - 10,$$
$$\log c = \log a - \log \sin A + 10 = 10 + \log a - \log \sin A.$$

HEIGHTS AND DISTANCES, ETC.

(1) In the year 1784, a base BC being measured on Blackheath, of a mile in length, the angles of elevation of Lunardi's balloon were taken, at the same time, by observers placed at its two extremities and in the middle; the one at B being $46^\circ 10'$, that at P $55^\circ 8'$, and that at C $54^\circ 30'$: required the height OA of the balloon.—Bonnycastle's *Trigonometry*.



Investigation.

By right-angled triangles,

$AB = OA \cot OBA$; $AP = OA \cot OPA$; $AC = OA \cot OCA$.

Now $AB^2 + AC^2 = 2AP^2 + 2BP^2$,

or $AB^2 + AC^2 - 2AP^2 = 2BP^2$,

hence by substitution we have

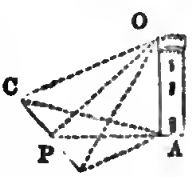
$$OA^2 \cot^2 OBA + OA^2 \cot^2 OCA - 2OA^2 \cot^2 OPA = 2BP^2,$$

$$OA^2 (\cot^2 OBA + \cot^2 OCA - 2 \cot^2 OPA) = 2BP^2;$$

$$\therefore OA^2 = \frac{2BP^2}{\cot^2 OBA + \cot^2 OCA - 2 \cot^2 OPA},$$

$$OA = \frac{BP\sqrt{2}}{\sqrt{\{\cot^2 OBA + \cot^2 OCA - 2 \cot^2 OPA\}}}.$$

(2) Observing an object OA at a distance, I took its angle of elevation OBA at the place where I stood, and found it to be $50^\circ 23'$; I then measured a distance PB of 60 yards, in the most convenient direction the ground afforded, and at this station found its elevation OPA to be $40^\circ 33'$; after which I measured on, in the



same line, 50 yards further to C, and at this place found its elevation OCA to be $30^\circ 49'$; from which it is required to determine the height of the object, and its distance from each of the three stations B, P, C.

Investigation.

Since $AB = OA \cot OBA$, $AP = OA \cot OPA$, $AC = OA \cot OCA$,

$\cos BPA = \dfrac{BP^2 + AP^2 - AB^2}{2AP.BP}$, and $\cos CAP$, which is equal to

$$-\cos BPA = -\frac{CP^2 + AP^2 - AC^2}{2AP.CP};$$

$$\therefore \frac{BP^2 + AP^2 - AB^2}{BP} = -\frac{CP^2 + AP^2 - AC^2}{CP},$$

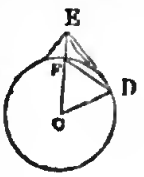
$$CP.AB^2 + BP.AC^2 - (CP+BP)\,AP^2 = (CP+BP).CP.BP;$$

$\therefore$ by substitution we have

$$CP.OA^2 \cot^2 OBA + BP.OA^2 \cot^2 OCA - (CP+BP) \times OA^2 \cot^2 OPA = (CP+BP).CP.BP,$$ this reduced gives

$$OA = \sqrt{\frac{CP \cot^2 OBA + BP \cot^2 OCA - (CP+BP) \cot^2 OPA}{(CP+BP).CP.BP}}.$$

(3) If the height of the mountain called the Peak of Teneriffe be $2\frac{1}{3}$ miles, as it is very nearly, and the angle taken at the top of it, as formed between a plumb-line and a line conceived to touch the earth in the horizon, or farthest visible point, be $88^\circ 2'$; it is required from these measures to determine the magnitude of the whole earth, and the utmost distance that can be seen on its surface from the top of the mountain, supposing the form of the earth to be perfectly globular?



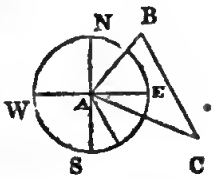
Let EF represent the height of the mountain, ED the tangent $\angle E = 88^\circ 2'$, DC a semidiameter of the earth.

Since EDC is a right angle, and $\angle C = 1^\circ 58'$, $\angle CDF = \angle CFD$, each of which $= 89^\circ 1'$, and $\angle EFD = 90^\circ 59'$, consequently $\angle EDF = 0^\circ 59'$.

Then, as sin $\angle EDF$ 0° 59′ ..... log ar. co. 9·765443
: sin $\angle EFD$ 90° 59′ ..... log 9·999936
:: EF $2\frac{1}{3}$ miles ..... log 0·367977
: ED 135·94 ..... log 2·133356

And (Euc. III. 36) $\frac{(135\cdot94)^2}{2\cdot3333} - 2\cdot3333 = 7916$ miles.

(4) A point of land was observed, by a ship at sea, to bear E. by S.; and after sailing N.E. 12 miles, it was found to bear S.E. by E. It is required to determine the place of that headland, and the ship's distance from it at the last observation.



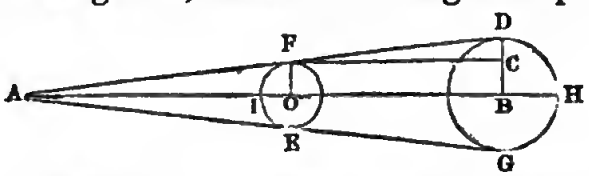
Let C be the point of land; A the former and B the latter place of observation. In the triangle ABC are given the angle A 56° 15′, the angle B 101° 15′, and the side AB 12 miles; consequently the angle C is 22° 30′.

Then, as sin 22°30′ $\angle C$ ..... log. ar. co. 10·417160
: sin 56° 15′ $\angle A$ ..... log 9·919846
:: 12 miles AB ..... log 1·079181

: 26·0728 miles BC, distance required, log 1·416187

(5) A string passes round the circumferences of two given circles, in the same plane, and at a given distance from each other: required the length of the string.

Let $GEIFDH$ represent the string touching the two circles, draw from the centres B and O the radii BD, OF to the points of tangence; these then being each perpendicular



to FD are parallel to one another; through F draw FC parallel to OB, then $FC = OB$ and $FO = BC$;

Let $OB = 14$ in., $OF = 1$ in., and $BD = 8\frac{1}{2}$ in.

Now $FD = \sqrt{FC^2 - DC^2} = \sqrt{OB^2 - (BD - OF)^2}$
$= \sqrt{(14^2 - 7\cdot5^2)} = 11\cdot82159$;

and sin $FCD = \frac{FD}{FC} = \frac{FD}{OB} = \frac{11\cdot82159}{14} = \cdot844399$;

$$\therefore \angle FCD = \angle OBC = \angle IOF = \angle 57^\circ 36' 25''.$$

$$\text{Hence } \angle DBH = 180^\circ - \angle DBO = 180^\circ - 57^\circ 36' 25'' = 122^\circ 23' 35''.$$

Then, as 360° : $57^\circ 36' 25''$ :: 6·2832 (circumference of the less circle) : 1·00543 = the arc FI. Also 360° : 122° 23′ 35″ :: 53·4072 (larger circum.) : 18·15742 = arc DH.

Hence $IF + FD + DH$ = 30·98444, twice which, or 61·96888 inches = 5 ft. 2 in. nearly, is the length of the band.

(6) The side of a hill forms an inclined plane whose angle of inclination is known; required the direction in which a rail or other road must run along the side of the hill, in order that it may have an ascent of 1 in every n feet.

(7) Wanting to know the height of an inaccessible tower; at the least distance from it, on the same horizontal plane, I took its angle of elevation equal to 58°; then going 300 feet directly from it, found the angle there to be only 32°; required its height, and my distance from it at the first station? Height 307·53, distance 192·15 ft.

(8) Being on a horizontal plane, and wanting to know the height of a tower placed on the top of an inaccessible hill: I took the angle of elevation of the top of the hill 40°, and of the top of the tower 51°; then measuring in a line directly from it to the distance of 200 feet farther, I found the angle at the top of the tower to be 33° 45′. What is the height of the tower? 93·33148 feet.

(9) From a window near the bottom of a house, which seemed to be on a level with the bottom of a steeple, I took the angle of elevation of the top of the steeple equal 40°; then from another window, 18 feet directly above the former, the like angle was 37° 30′: required the height and distance of the steeple? Height 210·44, distance 250·79 ft.

(10) Wanting to know the height of, and my distance from, an object on the other side of a river, which appeared to be on a level with the place where I stood, close by the side of the river; and not having room to measure backward, in the same line, because of the immediate rise of the bank, I placed a mark where I stood, and measured in a direction from the object, up the ascending ground, to the distance of 264 feet, where it was evident that I was above the level of

the top of the object; there the angles of depression were found to be, viz., of the mark left at the river's side 42°, of the bottom of the object 27°, and of its top 19°. Required the height of the object, and the distance of the mark from its bottom? Height 57·26 ft., distance 150·50 ft.

(11) Wanting to know my distance from an inaccessible object O, on the other side of a river, and having no instrument for taking angles, but only a chain or cord for measuring distances, from each of two stations, A and B, which were taken at 500 yards asunder, I measured in a direct line from the object O, 100 yards, viz., AC and BD each equal to 100 yards; also the diagonal AD measured 550 yards, and the diagonal BC, 560. What was the distance of the object from each station A and B? AO 536·81 yds., BO 500·47 yds.

(12) The elevation of a tower is observed. At a station (a) feet nearer, the elevation is the complement of the former; (b) feet nearer still, it is double the first elevation. Shew that the height is

$$\sqrt{\left\{(a+b)^2-\frac{a^2}{4}\right\}}.$$

(13) If the angles A, B, C of a triangle ABC be as the numbers 2, 3, 4 respectively; then

$$2\cos\frac{A}{2}=\frac{a+c}{b}.$$

(14) If in the three edges which meet at one angle of a cube, three points A, B, C be taken at distances a, b, c from the angle respectively; the area of the triangle ABC formed by joining the three points with each other

$$=\frac{1}{2}\sqrt{(a^2b^2+a^2c^2+b^2c^2)}.$$

(15) If the tangents of the angles of a plane triangle be in a geometric progression, whose ratio is n, shew that

$$\sin 2C = n \sin 2A.$$

(16) If the tangents of the semiangles of a plane triangle be in arithmetical progression, the cosines of the whole angles will also be in arithmetical progression. If R and r be the radii of the circles described about and inscribed in the triangle ABC, then the area of the triangle

$$\Delta = Rr(\sin A + \sin B + \sin C).$$

(17) In a right-angled triangle, CD, CE are drawn from the right angle C, making angles α, β with the hypothenuse; the area of the triangle

$$CDE = \frac{a^2b^2}{2c^2}(\cot\alpha \sim \cot\beta).$$

(18) If α, β, γ denote the lines bisecting the opposite sides a, b, c of any plane triangle, shew that

$$\alpha^4 + \beta^4 + \gamma^4 = \frac{9}{16}(a^4 + b^4 + c^4);$$

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = \frac{9}{16}(a^2b^2 + b^2c^2 + c^2a^2).$$

(19) Let a, b, c be the sides of a plane triangle, and α, β, γ the lines bisecting the angles and terminating in the opposite sides; then will

$$\frac{\alpha\beta\gamma}{abc} = \frac{4(a+b+c)}{(a+b)(b+c)(c+a)} \times \text{area of the triangle.}$$

(20) If lines be drawn from the angles of a plane triangle to a point within it, so as to make equal angles with each other, their sum will be represented by

$$\sqrt{\{a^2 - 2ab\cos(C + 60^0) + b^2\}}.$$

(21) The squares of the sides of a plane triangle added to the squares of the radii of the four circles respectively touching these sides is equal to four times the square of the circumscribing diameter; required a demonstration.

(22) If a, b, c be the sides of a triangle, and α, β, γ the perpendiculars upon them from the opposite angles, shew that

$$\frac{\alpha^2}{\beta\gamma} + \frac{\beta^2}{\alpha\gamma} + \frac{\gamma^2}{\alpha\beta} = \frac{bc}{a^2} + \frac{ac}{b^2} + \frac{ab}{c^2}.$$

(23) If R be the radius of the circle circumscribed about a given triangle ABC, and r the radius of the inscribed circle, shew that

$$\frac{1}{4}\cdot\frac{r}{R} = \frac{1}{2}\left\{1 - \left(\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2}\right)\right\}.$$

(24) In any plane triangle, the square of the distance

between the centre of the inscribed circle and the intersection of the perpendiculars

$$= 4R^2 - 2Rr - \frac{a^3 + b^3 + c^3}{a + b + c}.$$

(25) ABC is a plane triangle, P a point within it from which the sides subtend equal angles, it is required to find any number of such triangles having the sides, and the distances AP, BP, CP all expressed in whole numbers. (See fig. page 71).

(26) If $T(1)$ denote the sum of the angles of any polygon, $T(3)$ the sum of all the products that can be made by taking them by threes, $T(5)$ the sum of all the products that can be made by taking them by fives, and so on, then supposing radius to be unity

$$T(1) - T(3) + T(5) - T(7) + \&c. = 0.$$

(27) The sum of the diameters of the inscribed and circumscribing circles of any plane triangle is equal to

$$a \cot A + b \cot B + c \cot C.$$

(28) If a, b, c denote the sides and S the semiperimeter of any plane triangle, the square of the line drawn from the vertex to the lower extremity of the vertical diameter of the circumscribing circle is

$$\frac{bc\,(b + c)^2}{4S\,(S - a)}.$$

(29) In a plane triangle let r be the radius of the inscribed circle, R, R' and R'' the radius of the three circles touching the sides of the triangle externally; then will

$$\frac{1}{r} = \frac{1}{R} + \frac{1}{R'} + \frac{1}{R''},$$

required a demonstration.

(30) Let a, b, c be the three sides of a triangle, and r, r', r'', r''' the radii of the four circles which touch them, then

$$ab + ac + bc = rr' + rr'' + rr''' + r'r'' + r'r''' + r''r''',$$

that is, the sum of all the rectangles made by the binary combinations of the sides is equal to the sum of all the rect-

angles made by the binary combinations of the four radii of the circles of contact.

(31) If from one of the angles of a regular polygon of n sides, lines be drawn to all the angles, shew that their sum $= a \operatorname{cosec}^2 \frac{90^\circ}{n}$, $2a$ being the length of a side of the polygon.

(32) Having given three lines drawn from any point within a square to three of its angular points, determine a side of the square.

(33) The shadows of two vertical walls, which are at right angles to each other, and are a and a_1 feet in height, are observed when the sun is due south, to be b and b_1 feet in breadth; shew that if α be the sun's altitude, and β the inclination of the first wall to the meridian,

$$\cot \alpha = \sqrt{\left(\frac{b^2}{a^2} + \frac{b_1^2}{a_1^2}\right)}, \quad \cot \beta = \frac{ab_1}{a_1 b}.$$

(34) The distance between the centres of two wheels $= a$, and the sum of their radii $= c$; shew that the length of a string which crosses between the two wheels and just wraps round them

$$= 2\left\{\sqrt{(a^2 - c^2)} + c \cos^{-1}\left(-\frac{c}{a}\right)\right\}.$$

(35) If an equilateral triangle have its angular points in three parallel straight lines of which the middle one is distant from the outside ones by a and b, its side will be expressed by the following formula, viz.

$$\text{side} = 2\sqrt{\frac{a^2 + ab + b^2}{3}},$$

$$\text{or } 2\sqrt{\frac{(a+b)^2 - ab}{3}}:$$

required the proof.

(36) The hypothenuse (c) of a right-angled triangle ABC is trisected in the points D, E: prove that if CD, CE be joined, the sum of the squares of the sides of the triangle $CDE = \frac{2c^2}{3}$.

(37) Four places situated at unequal but given distances in the same straight line, appear to a spectator in the same plane with them to be at equal distances from each other; find the position of the spectator.

(38) The area of a quadrilateral inscribed in a circle
$$= \sqrt{\{(S-a)(S-b)(S-c)(S-d)\}}.$$

(39) If D and D' be the diagonals of a quadrilateral figure and ϕ their angle of intersection, then the
$$\text{area} = \frac{1}{2} D . D' \sin \phi.$$

CHAPTER V.

DEMOIVRE'S THEOREM.

33. This theorem, to which we give the name of the French geometer that discovered it, is as follows:

$$(\cos \phi + \sqrt{-1} \sin \phi)^n = \cos n\phi + \sqrt{-1} \sin n\phi \ldots\ldots \text{(A)}.$$

It expresses that, to raise the binomial
$$\cos \phi + \sqrt{-1} \sin \phi,$$
to any power whatever, it is sufficient to multiply the arc ϕ by the index of the power, we may either put the sign
$$+ \text{ or } - \text{ before } \sqrt{-1},$$
for that is the same as changing ϕ into $-\phi$.

The case where the index is a positive whole number is the first we shall proceed to consider.

By multiplication we find

$$(\cos \phi + \sqrt{-1} \sin \phi)(\cos \psi + \sqrt{-1} \sin \psi)$$
$$= \cos \phi \cos \psi - \sin \phi \sin \psi + \sqrt{-1} (\sin \phi \cos \psi + \cos \phi \sin \psi).$$

Now from the formulæ Art. 13, the real part of this product is equal to $\cos(\phi + \psi)$, and the imaginary part is equal to $\sqrt{-1} \sin(\phi + \psi)$; then

$$(\cos \phi + \sqrt{-1} \sin \phi)(\cos \psi + \sqrt{-1} \sin \psi)$$
$$= \cos(\phi + \psi) + \sqrt{-1} \sin(\phi + \psi).$$

That is to say, by multiplying together the two expressions of the form $\cos\phi + \sqrt{-1}\,\sin\phi$, we obtain a similar expression, which contains the sum of the two arcs.

In order to multiply the product by a new factor of the same form, it is sufficient to add the new arc to the other two; and thus we may continue the operation to any number of factors. Then, if we suppose that there are n factors, each equal to $\cos\phi + \sqrt{-1}\,\sin\phi$, it will become

$$(\cos\phi + \sqrt{-1}\,\sin\phi)^n = \cos n\phi + \sqrt{-1}\,\sin n\phi \ldots\ldots (1).$$

Let us consider the case where the exponent is fractional.

In replacing ϕ by $\frac{\phi}{n}$, the formula (1) becomes

$$\left(\cos\frac{\phi}{n} + \sqrt{-1}\,\sin\frac{\phi}{n}\right)^n = \cos\phi + \sqrt{-1}\,\sin\phi,$$

and, by extracting the nth root, and putting a fractional index in the place of the radical sign, the formula (A) will be demonstrated for the index $\frac{1}{n}$; for we have

$$(\cos\phi + \sqrt{-1}\,\sin\phi)^{\frac{1}{n}} = \cos\frac{\phi}{n} + \sqrt{-1}\,\sin\frac{\phi}{n} \ldots\ldots\ldots (2).$$

Generally, the expression $A^{\frac{m}{n}}$ signifies that we ought to raise A to the mth power, and afterwards take the nth root of the result. Consequently, if we raise $\cos\phi + \sqrt{-1}\,\sin\phi$ to the power m by the formula (1), and if afterwards we extract the nth root by the formula (2), it will become

$$(\cos\phi + \sqrt{-1}\,\sin\phi)^{\frac{m}{n}} = \cos\frac{m\phi}{n} + \sqrt{-1}\,\sin\frac{m\phi}{n} \ldots\ldots (3).$$

This is the formula (A) in which n is changed into any positive fraction $\frac{m}{n}$.

Finally, when the index is negative, we observe that

$$(\cos n\phi + \sqrt{-1}\,\sin n\phi)(\cos n\phi - \sqrt{-1}\,\sin n\phi)$$
$$= \cos^2 n\phi + \sin^2 n\phi = 1;$$

and from this we find,

$$\frac{1}{\cos n\phi + \sqrt{-1}\,\sin n\phi} = \cos n\phi - \sqrt{-1}\,\sin n\phi\,;$$

or which is the same thing, (by formula 1),

$$(\cos\phi + \sqrt{-1}\,\sin\phi)^{-n} = \cos(-n\phi) + \sqrt{-1}\,\sin(-n\phi) \ldots\ldots\ldots (4),$$

since the cosine of a positive arc is the same as the cosine of an equal arc taken negatively.

Thus, the formula (A) is true, when we take for n any positive or negative number whatever.

We have not used irrational indices seeing that they are not of any utility, as we cannot replace them by commensurable numbers, which however may differ therefrom as little as we please. And as to imaginary indices, they are not susceptible of any interpretation.

The formula (A), which is so simple and elegant, appears to have a serious defect when the index is fractional. In fact, the first member, being then equivalent to a radical expression, ought to have many values, whilst the second member presents only one. The explanations which follow have for their object to correct this imperfection.

Let us return to formula (2) in which n is a positive whole number. From the principles of Algebra, the first member, which is equivalent to $\sqrt[n]{(\cos\phi + \sqrt{-1}\,\sin\phi)}$, ought then to have n different values; and in order that the second may give all of them, we shall shew that it will suffice to replace ϕ by all the arcs which have the same sines and cosines as ϕ itself. The general expression of these arcs is $\phi + kC$, C designating the whole circumference, and k any whole positive or negative integer number. By putting $\phi + kC$ in the place of ϕ, the second member of formula (2) becomes,

$$\cos\frac{\phi + kC}{n} + \sqrt{-1}\,\sin\frac{\phi + kC}{n} \ldots\ldots\ldots\ldots (5).$$

And in this state, we say that it has precisely the same values as the first member.

And first, since n is an integer, it is clear, from formula (1), that by raising this second member to the power n, we arrive again at the form $\cos\phi + \sqrt{-1}\,\sin\phi$.

In the second place, if we make successively $k=0$, $k=1$, $k=2$ ..... $k=n-1$, we obtain n different values.

In fact, let there be any two values,

$$\cos\frac{\phi+\alpha C}{n}+\sqrt{-1}\,\sin\frac{\phi+\alpha C}{n},$$

$$\text{and }\cos\frac{\phi+\beta C}{n}+\sqrt{-1}\,\sin\frac{\phi+\beta C}{n};$$

in which α and β are integer numbers $<n$. In order that they may be equal, it is necessary to have separately, an equality between the real and the imaginary parts; then the difference of the two arcs $\frac{\phi+\alpha C}{n}$ and $\frac{\phi+\beta C}{n}$ ought to be equal to one or more circumferences: but that difference, which is $\frac{(\alpha-\beta)\,C}{n}$, is *less* than C, seeing that α and β are $<n$: consequently the two values are unequal.

In the third place, if we take for k other numbers than 0, 1, 2 ..... $n-1$, we shall find new values. In fact, all the other positive or negative integer numbers may be represented by the formula $n\gamma+n'$, γ being any integer number, positive or negative, and n' a positive integer $<n$; moreover by making $k=n\gamma+n'$ the expression (5) becomes

$$\cos\left(\gamma C+\frac{\phi+n'C}{n}\right)+\sqrt{-1}\,\sin\left(\gamma C+\frac{\phi+n'C}{n}\right),$$

or rather, suppressing the useless circumferences,

$$\cos\frac{\phi+n'C}{n}+\sqrt{-1}\,\sin\left(\frac{\phi+n'C}{n}\right);$$

and as n' is a positive number $<n$, that value is comprehended under those we have obtained by putting

$$k=0,\ 1,\ 2\ \ldots\ldots\ n-1.$$

Thus the second member of the formula (2) will acquire the generality, which it ought to have, if we take care to have for the arc ϕ not only the arc ϕ itself, but also the arcs

$$\phi+C,\ \phi+2C,\ \ldots\ldots\ \phi+(n-1)C.$$

Formula (3), ought also to be interpreted in an analogous manner, as we find by raising $\cos\phi+\sqrt{-1}\,\sin\phi$ to the

mth power, and extracting the nth root of the result. The formula (1), relating to the case of a whole positive index, gives

$$(\cos\phi + \sqrt{-1}\sin\phi)^m = \cos m\phi + \sqrt{-1}\sin m\phi,$$

and afterwards, for the nth root, the formula (2) gives

$$(\cos\phi + \sqrt{-1}\sin\phi)^{\frac{m}{n}} = \cos\frac{m\phi}{n} + \sqrt{-1}\sin\frac{m\phi}{n}.$$

But in order that the second member may have the same extension as the first, we must, after what we have already explained, put $m\phi + kC$ in the place of $m\phi$; or, which is the same thing, replace $m\phi$ successively by

$$m\phi,\ m\phi + C,\ \ldots\ldots\ m\phi + (n-1)\,C.$$

34. Formulæ for $\sin n\phi$, $\cos n\phi$, and $\tan n\phi$. Referring back to Demoivre's formulæ,

$$\cos n\phi + \sqrt{-1}\sin n\phi = (\cos\phi + \sqrt{-1}\sin\phi)^n \ldots\ldots\ldots (1),$$

in which we suppose n to be a positive integer. Changing ϕ into $-\phi$ it becomes

$$\cos n\phi - \sqrt{-1}\sin n\phi = (\cos\phi - \sqrt{-1}\sin\phi)^n \ldots\ldots (2).$$

Adding and subtracting (1) and (2),

$$\cos n\phi = \frac{(\cos\phi + \sqrt{-1}\sin\phi)^n + (\cos\phi - \sqrt{-1}\sin\phi)^n}{2} \ldots (3),$$

$$\sin n\phi = \frac{(\cos\phi + \sqrt{-1}\sin\phi)^n - (\cos\phi - \sqrt{-1}\sin\phi)^n}{2} \ldots (4).$$

Expanding by the binomial theorem, and suppressing the terms which destroy each other, we obtain

$$\cos n\phi = (\cos\phi)^n - \frac{n(n-1)}{1.2}(\cos\phi)^{n-2}(\sin\phi)^2 + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}(\cos\phi)^{n-4}(\sin\phi)^4 - \&c. \ldots (5),$$

$$\sin n\phi = n(\cos\phi)^{n-1}\sin\phi - \frac{n(n-1)(n-2)}{1.2.3}(\cos\phi)^{n-3}(\sin\phi)^3 + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}(\cos\phi)^{n-5}(\sin\phi)^5 - \&c. \ldots (6).$$

These formulæ express the sine and cosine of the multiple $n\phi$ in terms of the sine and cosine of the simple arc ϕ: the law of the terms is evident, and in the same manner as the binomial formulæ from which they are derived, the terms must be continued until we find a term equal to nothing.

To find $\tan n\phi$, we must divide (6) by (5),

$$\frac{\sin n\phi}{\cos n\phi} = \tan n\phi = \frac{n\tan\phi - \frac{n(n-1)(n-2)}{1.2.3}\tan^3\phi - \&c.}{1 - \frac{n(n-1)}{1.2}\tan^2\phi + \&c.}.$$

35. Development of the sine and cosine in a series.

From equations (5) and (6), Euler has deduced the series which expresses the sine and cosine of an arc in terms of the arc itself.

Retaining the supposition that n is a whole number, we can dispose of ϕ in such a manner that $n\phi$ may be equal to any arc x; putting then $n\phi = x$, we have $n = \frac{x}{\phi}$, and consequently the equations (5) and (6) will become

$$\cos x = (\cos\phi)^n - \frac{x(x-\phi)}{1.2}(\cos\phi)^{n-2}\left(\frac{\sin\phi}{\phi}\right)^2 + \frac{x(x-\phi)(x-2\phi)(x-3\phi)}{1.2.3.4}(\cos\phi)^{n-4}\left(\frac{\sin\phi}{\phi}\right)^4 - \&c. \ldots (1),$$

$$\sin x = x(\cos\phi)^{n-1}\left(\frac{\sin\phi}{\phi}\right) - \frac{x(x-\phi)(x-2\phi)}{1.2.3}(\cos\phi)^{n-3}\left(\frac{\sin\phi}{\phi}\right)^3 + \frac{x(x-\phi)(x-2\phi)(x-3\phi)(x-4\phi)}{1.2.3.4.5}(\cos\phi)^{n-5}\left(\frac{\sin\phi}{\phi}\right)^5 - \&c. \ldots (2).$$

Let us now conceive ϕ to diminish to zero and n to increase to infinity, then the above formulæ will preserve no trace either of ϕ or of n, and will contain the arc x only.

When ϕ becomes zero, we have

$$\cos\phi = 1 \text{ and also } \frac{\sin\phi}{\phi} = 1.$$

Admitting that the powers of $\cos\phi$ and of $\frac{\sin\phi}{\phi}$ be also equal to unity, however great the indices may be, the above formulæ become

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \&c. \ldots (3),$$

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \&c. \ldots (4).$$

The following very sensible observations on these series are made by a correspondent of the *Mechanics' Magazine.* (See No. 1256, pages 231, 232).

"Now to many students, who are not altogether without thinking what they are about, and yet have not attained to clear notions on the distinction between *things themselves* and the *numerical measures* of those things—the above series are a sad stumbling-block, and well they may be. The reader who has these misty notions, naturally asks—'How can a sine or cosine, which I know to be a number or ratio, be expressed in terms of *an angle*, which I conceive to be an *opening*?' How indeed! About the *ne plus ultra* of all conceivable absurdities would be the following equation; sine of a certain opening

$$= \text{that opening} - \frac{1}{6}(\text{cube of that opening}) + \&c.$$

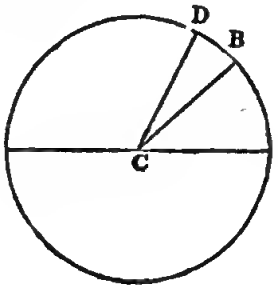
"The absurdity arises from considering the symbol (ϕ) as standing for the Euclidian angle or opening, instead of for a certain numerical fraction or ratio which *measures* that angle, and which bears the same proportion to *other fractions* (the measures of other angles), which the Euclidian angles themselves do. Cos ϕ is really an abridgment for 'cosine of that Euclidian angle the length of whose arc bears the same proportion to the length of the radius, as the number ϕ does to unity.'

"Again, the *use* of these series is seldom seen, and because the series on the left hand side of the equation is infinite in the number of its terms, the student is apt sometimes to conceive that an immense number of those terms must be taken to get anything like a tolerably near value for cos ϕ or sin ϕ. To obviate this fancy, and at the same time to shew clearly the real nature of the equation, let us take a numerical example and work it out. Suppose, then, we endeavour to find the numerical value of the cosine of 30 degrees by means of the series. Here (ϕ) means

$$\frac{\text{length of arc subtending the angle of } 30^0}{\text{length of radius}},$$

and this is the first thing to find.

"Now, we know that the length of the circumference of a circle bears the proportion to its diameter of (π) to unity: π being nearly equal to 3·1416. We have then



$$\frac{\text{arc } AB}{\frac{1}{2}\pi} = \frac{30^0}{90^0} = \frac{1}{3};$$

$\therefore$ arc $AB = \frac{1}{3} \times$ a quarter of the circumference, $= \dfrac{\text{semi-circumference}}{6}$;

$$\therefore \frac{\text{arc } AB}{\text{radius}} = \frac{1}{6} \cdot \frac{\text{semi-circumference}}{\text{radius}} = \frac{1}{6}\pi = \frac{3{\cdot}1416}{6},$$

so that here (ϕ) is equal to ·5236.

"Take only the first three terms of the series for cos ϕ, so that

$$\cos 30^0 = 1 - \frac{1}{2}(\cdot 5236)^2 + \frac{1}{24}(\cdot 5236)^4.$$

Keep only four places of decimals in multiplying, and the second side becomes

$$= 1 - 0{\cdot}137078 + 0{\cdot}003130 = \cdot 866052.$$

"Now turn to a table of natural sines, &c., and you will find the cosine of 30° to be ·8660254, which agrees with that just found from the series up to the fifth place of decimals, and differs less from it than the cosines of 30° 1′, or 29° 59′, differ from the cosine of 30°."

36. In the applications of the higher branches of mathematics we often find it necessary to express $(\sin \phi)^n$ and $(\cos \phi)^n$ in terms of the sine and cosine of the simple arc. We proceed in the following manner:

$$\text{Let } \cos \phi + \sqrt{-1} \sin \phi = x,$$

$$\text{then } \cos \phi - \sqrt{-1} \sin \phi = \frac{1}{x}.$$

$$\text{By addition, } 2\cos\phi = x + \frac{1}{x},$$

$$\text{and by subtraction, } 2\sqrt{-1}\sin\phi = x - \frac{1}{x};$$

$$\therefore 2^n(\cos\phi)^n = \left(x + \frac{1}{x}\right)^n$$

$$= x^n + nx^{n-2} + n\left(\frac{n-1}{2}\right)x^{n-4} + \ldots \frac{n}{x^{n-2}} + \frac{1}{x^n}$$

= (collecting into pairs the terms equidistant from each extremity of the series)

$$\left(x^n + \frac{1}{x^n}\right) + n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + n\left(\frac{n-1}{2}\right)\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \&c.$$

but $\{\cos\phi + \sqrt{-1}\sin\phi\}^n = \cos n\phi + \sqrt{-1}\sin n\phi = x^n$,

$$\{\cos\phi - \sqrt{-1}\sin\phi\}^n = \cos n\phi - \sqrt{-1}\sin n\phi = \frac{1}{x^n},$$

by adding and subtracting, we have

$$2\cos n\phi = x^n + \frac{1}{x^n}, \text{ and } 2\sqrt{-1}\sin n\phi = x^n - \frac{1}{x^n};$$

$$\therefore \left(x^n + \frac{1}{x^n}\right) + n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + n\frac{n-1}{2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \&c.$$

$$= 2\cos n\phi + 2n.\cos(n-2)\phi + 2n\frac{n-1}{2}\cos(n-4)\phi, \&c.;$$

$$\therefore 2^{n-1}(\cos\phi)^n = \cos n\phi + n\cos(n-2)\varphi + n\frac{n-1}{2}\cos(n-4)\phi, \&c.;$$

the last term of the series being

$$\frac{1}{2}n\left(\frac{n-1}{2}\right)\ldots\frac{n-\frac{1}{2}n+1}{\frac{1}{2}n}, \text{ when } n \text{ is even,}$$

and $n\left(\frac{n-1}{2}\right)\ldots\frac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)}\cos\phi$, when n is odd.

From what has been given above, we have for the sine

$$2\sqrt{-1}\sin\phi = x - \frac{1}{x}, \text{ and } 2\sqrt{-1}\sin n\phi = x^n - \frac{1}{x^n},$$

$$(2\sqrt{-1}\,\sin\phi)^n = \left(x - \frac{1}{x}\right)^n = \left\{x + \left(-\frac{1}{x}\right)\right\}^n$$

$$= x^n - nx^{n-2} + \frac{n(n-1)}{2}\,x^{n-4} \ldots \frac{(-1)^n}{x^n} \quad \ldots\ldots\ldots\ldots (a)$$

$$= \left\{x^n + \left(\frac{-1}{x}\right)^n\right\} - n\left\{x^{n-2} + \left(\frac{-1}{x}\right)^{n-2}\right\}$$

$$+ \frac{n(n-1)}{2}\left\{x^{n-4} + \left(\frac{-1}{x}\right)^{n-4}\right\} \ldots\ldots\ldots\ldots (b),$$

when n is even the series (b) becomes

$$(-1)^{\frac{n}{2}}\,2^n \sin^n\phi = \left(x^n + \frac{1}{x^n}\right) - n\left(x^{n-2} + \frac{1}{x^{n-2}}\right)$$

$$+ n\,\frac{n-1}{2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) - \ldots$$

$$+ (-1)^{\frac{n}{2}}\,n\,\frac{n-1}{2} \ldots \frac{n - \frac{1}{2}n + 1}{\frac{1}{2}n}$$

$$= 2\cos n\phi - n \,.\, 2\cos(n-2)\,\phi + n\,\frac{n-1}{2}\,2\cos(n-4)\,\phi -$$

$$\ldots\ldots + (-1)^{\frac{n}{2}}\,n\,\frac{n-1}{2} \ldots \frac{n - \frac{1}{2}n + 1}{\frac{1}{2}n};$$

$$\therefore\ (-1)^{\frac{n}{2}}\,2^{n-1}\sin^n\phi = \cos n\phi - n\,\cos(n-2)\,\phi$$

$$+ n\,\frac{n-1}{2}\cos(n-4)\,\phi - \ldots$$

$$\ldots\ldots + (-1)^{\frac{n}{2}}\,\tfrac{1}{2}n\,\frac{n-1}{2} \ldots \frac{n - \frac{1}{2}n + 1}{\frac{1}{2}n}.$$

If n be odd, then $(\sqrt{-1})^n = \sqrt{-1}\,(\sqrt{-1})^{n-1} = (-1)^{\frac{n-1}{2}}\sqrt{-1}$; and the last term of (b) is

$$(-1)^{\frac{1}{2}(n-1)}\,n\,\frac{n-1}{2} \ldots \frac{n - \frac{1}{2}(n-1) + 1}{\frac{1}{2}(n-1)}\left(x - \frac{1}{x}\right).$$

Hence, when n is an odd number (b) becomes

$$(-1)^{\frac{n-1}{2}} 2^n . \sqrt{-1} \sin^n \phi = \left(x^n - \frac{1}{x^n}\right) - n\left(x^{n-2} - \frac{1}{x^{n-2}}\right)$$

$$+ n\frac{n-1}{2}\left(x^{n-4} - \frac{1}{x^{n-4}}\right) - \ldots$$

$$\ldots\ldots (-1)^{\frac{1}{2}(n-1)} n \frac{n-1}{2} \ldots \frac{n - \frac{1}{2}(n-1) + 1}{\frac{1}{2}(n-1)} \left(x - \frac{1}{x}\right),$$

$$= 2\sqrt{-1} \sin n\phi - n \cdot 2\sqrt{-1} \sin (n-2)\phi + \ldots$$

$$\ldots\ldots + (-1)^{\frac{n-1}{2}} n \frac{n-1}{2} \ldots \frac{n - \frac{1}{2}(n-1) + 1}{\frac{1}{2}(n-1)} 2\sqrt{-1} \sin \phi;$$

$$\therefore (-1)^{\frac{n-1}{2}} 2^{n-1} \sin^n \phi = \sin n\phi - n \sin (n-2)\phi$$

$$+ n\frac{n-1}{2} \sin (n-4)\phi - \ldots$$

$$\ldots\ldots + (-1)^{\frac{n-1}{2}} n \frac{n-1}{2} \ldots \frac{n - \frac{1}{2}(n-1) + 1}{\frac{1}{2}(n-1)} \sin \phi.$$

(1) Sum the series

$$\cos \phi + \cos 2\phi + \cos 3\phi + \&c. + \cos n\phi.$$

The sum is equal to the two following series,

$$\frac{1}{2}(x + x^2 + x^3 + \&c. + x^n),$$

$$\frac{1}{2}\left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \&c. + \frac{1}{x^n}\right);$$

$$\therefore = \frac{1}{2}\left(\frac{x^{n+1} - x}{x-1} + \frac{(x^n - 1)}{x^n (x-1)}\right)$$

$$= \frac{1}{2}\frac{(x^n - 1)(x^{n+1} + 1)}{x^n (x-1)} = \frac{1}{2}\frac{\left(x^{\frac{n}{2}} - \frac{1}{x^{\frac{n}{2}}}\right)\left(x^{\frac{n+1}{2}} + \frac{1}{x^{\frac{n+1}{2}}}\right)}{x^{\frac{1}{2}} - \frac{1}{x^{\frac{1}{2}}}}.$$

$$\text{Now, } \frac{x^{\frac{n}{2}} - \frac{1}{x^{\frac{n}{2}}}}{\sqrt{x} - \frac{1}{\sqrt{x}}} = \sqrt{\left\{\frac{x^{\frac{n}{2}} - \frac{1}{x^{\frac{n}{2}}}}{\sqrt{x} - \frac{1}{\sqrt{x}}}\right\}^2}$$

$$= \sqrt{\frac{\left(x^{\frac{n}{2}} + \frac{1}{x^{\frac{n}{2}}}\right)^2 - 4}{\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 - 4}} = \sqrt{\frac{\left(2\cos\frac{n\phi}{2}\right)^2 - 4}{\left(2\cos\frac{\phi}{2}\right)^2 - 4}}$$

$$= \sqrt{\frac{1 - \cos^2\frac{n\phi}{2}}{1 - \cos^2\frac{n\phi}{2}}} = \frac{\sin\frac{n\phi}{2}}{\sin\frac{n\phi}{2}};$$

consequently the sum of the series is equal to

$$\frac{\sin\frac{n\phi}{2}}{\sin\frac{n\phi}{2}} \times \cos\left(\frac{n+1}{2}\right)\phi.$$

(2) Sum the series $\operatorname{cosec}\phi + \operatorname{cosec} 2\phi \operatorname{cosec} 2^2\phi$ to n terms.

$$\operatorname{cosec}\phi = \cot\frac{\phi}{2} - \cot\phi,$$

$$\operatorname{cosec} 2\phi = \cot\phi - \cot 2\phi,$$

&c.

$$\operatorname{cosec} 2^{n-1}\phi = \cot 2^{n-2}\phi - \cot 2^{n-1}\phi;$$

hence, by addition,

$$\operatorname{cosec}\phi + \operatorname{cosec} 2\phi + \&c.\ \operatorname{cosec} 2^{n-1}\phi = \cot\frac{\phi}{2} \cot 2^{n-1}\phi.$$

(3) Sum the series

$$\frac{1}{2}\tan\frac{\phi}{2} + \frac{1}{2^2}\tan\frac{\phi}{2^2} + \frac{1}{2^3}\tan\frac{\phi}{2^3} + \&c. \text{ to } n \text{ terms.}$$

Since $\tan\frac{\phi}{2} = \cot\frac{\phi}{2} - 2\cot\phi,$

we have $\frac{1}{2}\tan\frac{\phi}{2} = \frac{1}{2}\cot\frac{\phi}{2} - \cot\phi$,

$$\frac{1}{2^2}\tan\frac{\phi}{2^2} = \frac{1}{2^2}\cot\frac{\phi}{2^2} - \frac{1}{2}\cot\frac{\phi}{2},$$

&c.....

$$\frac{1}{2^n}\tan\frac{\phi}{2^n} = \frac{1}{2^n}\cot\frac{\phi}{2^n} - \frac{1}{2^{n-1}}\cot\frac{\phi}{2^{n-1}}.$$

By addition we have the sum

$$= \frac{1}{2^n}\cot\frac{\phi}{2^n} - \cot\phi.$$

(4) Sum the series $\sin\phi + \sin 2\phi + \sin 3\phi +$ &c.

MISCELLANEOUS EXAMPLES.

(1) The square of the distance between the centres of two of the escribed circles of a triangle exceeds the square of the sum of their radii, by square of the opposite side of the triangle.

(2) Let a, b, c be the three distances of a point from three successive angles of a square field, then if S be the side of the square, and Δ the area of the triangle, whose sides are $a, b\sqrt{2}, c$, it is required to prove that

$$S^2 = \frac{1}{2}(a^2 + c^2) \pm 2\Delta.$$

(3) The continued product of the radii of any three circles mutually touching each other, divided by the radius of the circle passing through their points of contact, gives the area of the triangle formed by joining their centres.

(4) Shew that if there be inscribed in a circle a regular figure, each of whose sides is an m^{th} part of the radius, the secant of the angle at the centre subtending each side

$$= \frac{2m^2}{2m^2 - 1}.$$

(5) Two regular polygons of the same number of sides being described, the one within, and the other without, the

same circle, what will be the number of sides, when the whole space intercepted *between* the two polygonal boundaries is an assigned part of either polygon? Ex. What is the figure when the exterior area is $\frac{1}{3}$ of the interior polygon?

(6) In a right-angled triangle, if the hypothenuse (c) be divided into segments (x), (y) by the line which bisects the right angle, and t = the tangent of half the difference of the acute angles,

$$x : y :: 1 + t : 1 - t.$$

(7) To divide a given angle into two others, whose sines shall be in the ratio of $m : n$.

(8) A circle is inscribed in an equilateral triangle, an equilateral triangle in the circle, a circle again in the latter triangle, and so on; if r, r_1, r_2, r_3, &c., be the radii of the circles, shew that

$$r = r_1 + r_2 + r_3 + \&c.$$

(9) If a, b, c be the sides of a triangle, and p, q, r perpendiculars from a point within the triangle bisecting the sides, prove that

$$4\left(\frac{a}{p} + \frac{b}{q} + \frac{c}{r}\right) = \frac{abc}{pqr}.$$

(10) It is required to determine the continued product of n terms of the series

$$(\sin\phi \cos\tfrac{1}{2}\phi)^{\frac{1}{2}} (\sin\tfrac{1}{2}\phi \cos\tfrac{1}{4}\phi)^{\frac{1}{4}} (\sin\tfrac{1}{4}\phi \cos\tfrac{1}{8}\phi)^{\frac{1}{8}}$$

$$\ldots\ldots (\sin\tfrac{1}{2^{n-1}}\phi \cos\tfrac{1}{2^n}\phi)^{\frac{1}{2^n}}.$$

(11) The circumference of the inner of two concentric circles, of which R and r are the radii, is divided into n equal parts; shew that the sum of the squares of the lines drawn from the points of division to any point in the circumference of the outer circle $= n(R^2 + r^2)$.

(12) The ratio between the area of an equilateral and equiangular decagon described about a circle, and that of another within the same circle, is equal to

$$\cfrac{8}{7+\cfrac{1}{4+\cfrac{1}{4+\ldots\ldots}}}$$

(13) In a plane triangle ABC, having given the sum of the sides AC, CB, the perpendicular from the vertex C upon the base AB; and the difference of the segments of the base made by the perpendicular; find the sides of the triangle.

(14) The length of a road in which the ascent is 1 foot in 5, from the foot of a hill to the top is a mile and two-thirds. What will be the length of a zigzag road, in which the ascent is 1 foot in 12?

(15) At each end of a horizontal base measured in a known direction from the place of an observer, the angle which the distance of the other end and a certain object subtends is observed, as also the angle of elevation of the object at one end of the base. Find its height and bearing.

(16) A staff one foot long stands on the top of a tower 200 feet high. Shew that the angle which it subtends at a point in the horizontal plane 100 feet from the base of the tower is nearly $6'{\cdot}51''$.

(17) Find the degrees, minutes, and seconds, in the angle whose circular measure is $\cdot 1$; also the values of $\cos m\pi$ and $\tan^{-1}(-1)^m$ (m an integer).

(18) Two circles have a common radius (r) and a circle is described touching this radius and the two circles: prove that the radius of the circle which touches the three $= \frac{3r}{56}$.

(19) At noon a column in the E.S.E. cast upon the ground a shadow, the extremity of which was in the direction N.E.: the angle of elevation of the column being a°, and the distance of the extremity of the shadow from the column (a) feet, determine the height of the column.

(20) $\mathrm{Tan}^{-1}\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) - \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$.

(21) If A, B, C be the angles of a triangle,

$$\cos nA + \cos nB + \cos nC = 1 \pm 4 \sin \frac{nA}{2} \sin \frac{nB}{2} \sin \frac{nC}{2},$$

$$\text{or } = -1 \pm 4 \cos \frac{nA}{2} \cos \frac{nB}{2} \cos \frac{nC}{2},$$

according as n is odd or even.

(22) In a regular polygon of n sides inscribed in a circle whose radius is (r), if a be the distance of any point from the centre, and perpendiculars be drawn from this point upon all the sides of the polygon, the sum of the squares of the lines joining the feet of the adjacent perpendiculars is

$$n \sin^2 \frac{2\pi}{n} (r^2 + a^2).$$

(23) Four objects situated at unequal but given distances in the same straight line, appear to a spectator in the same plane with them to be at equal distances from each other, it is required to determine his position.

(24) If a', b' be the segments of the hypothenuse made by a line bisecting the right angle, then

$$\frac{a'b'}{ab} = \frac{a^2 + b^2}{(a+b)^2}.$$

(25) A boy flying a kite at noon, when the wind was blowing α^0 from the south, and the angular distance of the kite's shadow from the north was β^0, the wind suddenly changed to $\alpha_1{}^0$ from the south, and the shadow to $\beta_1{}^0$ from the north, and the kite was raised as much above 45^0 as it had before been below that elevation. Shew that θ^0 being the angular elevation of the sun, and $45^0 - \phi^0$ that of the kite at first,

$$\tan^2 \theta^0 = \frac{\sin \beta \sin \beta}{\sin (\alpha - \beta) \sin (\alpha_1 - \beta_1)},$$

$$\tan^2 (45^0 - \phi^0) = \frac{\sin (\alpha - \beta) \sin \beta_1}{\sin (\alpha_1 - \beta_1) \sin \beta}.$$

FINIS.

ERRATA.

Page 42, line 8, for $\frac{3}{2}$ read $\frac{3}{6}$.

... 64 ... 4, the thick and thin line to be transposed.

... 96 Supply *B* in fig. 2, as in fig. 1.

THE ELEMENTS

OF

SPHERICAL TRIGONOMETRY.

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JOHN WEALE, 59, HIGH HOLBORN.

1849.

ERRATA TO THE *PLANE* TRIGONOMETRY.

Page 10, Line 5, *for* AP *read* AP^2.

„ 9, *for* DP *read* DP^2.

53, „ 3 from top, *for* $\sin 2A$ *read* $\cos 2A$.

10, *for* $\cos \frac{A}{2}$ *read* $\sqrt{2}.\cos \frac{A}{2}$.

„ 11 *for* $\operatorname{cosec}^2 A$ *read* $\operatorname{cosec}^2 2A$.

„ 12 *for* $-\sqrt{-1}$ *read* $-\sqrt{-1}.\sin A$.

77, Line 24, *for* $\frac{\sqrt{3}}{2}$ *read* $\frac{\sqrt{3}}{4}$.

93, „ 2, *for* $24r\frac{180}{24}$ *read* $24r\sin\frac{180}{24}$.

100, „ 3, *for* $ac(\theta+\varphi)$ *read* $ac\sin(\theta+\varphi)$.

102, Fig. 2, *for* C *read* D; *and for* D *read* C.

103, Line 5 from bottom, *for* $c\cos\beta$ *read* $c\cos\gamma$.

PREFACE.

In the compilation of this work, the most esteemed writers, both English and foreign, have been consulted, but those most used are De Fourcy and Legendre.

Napier's Circular Parts have been treated in a manner somewhat different to most modern writers. The terms conjunct and adjunct, used by Kelly and others, are here retained, as they appear to be more conformable to the practical views of Napier himself.

There are many other parts connected with Spherics that might be treated of, but which are not adapted to a Rudimentary Treatise like the present; those, however, who wish to see all the higher departments fully developed, must consult the writings of that distinguished mathematician, Professor Davies, of the Royal Military Academy, Woolwich.

Hutton's Course, the Ladies' and Gentleman's Diaries, (latterly comprised in one), Leybourne's Repository, the Mechanics' Magazine, and various other periodicals, teem with the productions of his fertile mind, both on this and other kindred subjects.

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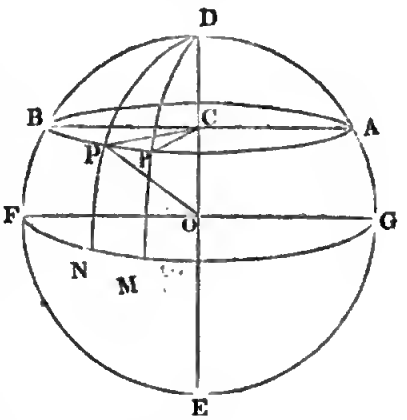
SPHERICS.

PRELIMINARY CHAPTER.

1. A SPHERE is a solid determined by a surface of which all the points are equally distant from an interior point, which is called the centre of the sphere.

2. Every section of a sphere made by a plane cutting it is the arc of a circle.



Let O be the centre of the sphere, APBA a section made by a plane passing through it, draw OC to the cutting plane, and produce it both ways to D and E, and draw the radii of the sphere OA, OP.

Now, since OCP and OCA are right angles, $OA^2 - OC^2 = AC^2$, and $OP^2 - OC^2 = PC^2$, but $OA^2 = OP^2$; $\therefore AC^2 = PC^2$ or AC = PC; hence the section APBA is a circle.

If the cutting plane pass through the centre, the radius of the section is evidently equal to the radius of the sphere, and such a section is called a great circle of the sphere.

3. The poles of any circle are the two extremities of that diameter or axis of the sphere which is perpendicular to the plane of that circle; and therefore either pole of any circle is equidistant from every part of its circumference, and, if it be a great circle, its pole is 90° from the circumference. A spherical triangle is the portion of space comprised between three arcs of intersecting great circles.

4. The angles of a spherical triangle are those on the surface of the sphere contained by the arcs of the great circles which form the sides, and are the same as the inclinations of the planes of those great circles to one another.

5. Any two sides of a spherical triangle are greater than the third side.

Since by Euclid XI. 20, any two of the plane angles, which form the solid angle at O, are together greater than the third, hence any two of the arcs which measure those angles must be greater than the third.

6. Since the solid angle at O (see fig. p. 3) is contained by three plane angles, and by Euclid XI. 21, these are together less than four right angles, hence the three arcs of the spherical triangle which measure those angles must be together less than the circumference of a great circle, that is $a + b + c > 360$, and since any two sides of a triangle is greater than the third, we have $a + b > c$; $b + c > a$; $a + c > b$.

ON THE POLAR OR SUPPLEMENTAL TRIANGLE.

7. If three arcs of great circles be described from the angular points A, B, C, of any spherical triangle A B C, as poles, the sides and angles of the new triangle, D F E, so formed will be the supplements of the opposite angles and sides of the other, and *vice versâ*.

Since B is the pole of D F, then B D is a quadrant, and since C is the pole of D E, C D is a quadrant; therefore the distances of the points B and C from D being each a quadrant, they are equal to each other, hence D is the pole of B C.

DE=180°−C; EF=180°−A;

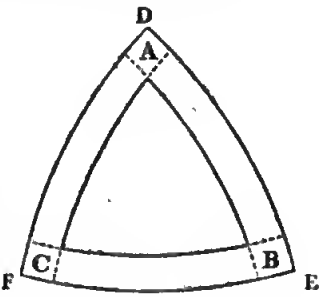
FD=180°−B; and D=180°−BC;

E=180°−AC; F=180°−AB.

Also, AB=180°−F; BC=180°−D;

AC=180°−E; A=180−FE;

B=180°−FD; C=180°−DE.



The sum of the three angles of a spherical triangle is greater than two right angles, and less than six right angles.

For if $a' + b' + c'$ be the sides of the supplemental or polar triangle, A=180°−a'; B=180°−b'; C=180°−c';

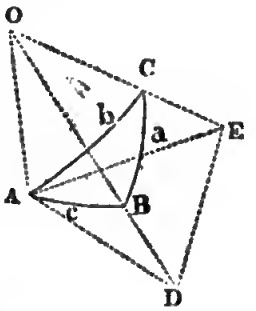
hence A+B+C+a'+b'+c'=6×90=6 right angles;

but $a' + b' + c'$ is less than four right angles, by Euclid XI. 21; therefore A+B+C is greater than two right angles; and as the sides a', b', c', of the polar triangle must have some magnitude, the sum of the three angles A, B, C must be less than six right angles.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

8. Spherical Trigonometry treats of the various relations between the sines, tangents, &c., of the known parts of a spherical triangle, and those that are unknown; or, which is the same thing, it gives the relations between the parts of a solid angle formed by the inclination of three planes which meet in a point, for the solid angle is composed of six parts, the inclinations of the three plane faces to each other, and also the inclinations of the three edges; in fact, a work might be written on this subject without using the spherical triangle at all, for the six parts of the spherical triangle are measures of the six parts of the solid angle at O. See fig.



9. If a spherical triangle have one of its angles a right angle, it is called a right-angled triangle; if one of its sides be a quadrant, it is called a quadrantal triangle; if two of the sides be equal, it is called an isosceles triangle, &c., as in Plane Trigonometry.

10. To determine the sines and cosines of a spherical triangle in terms of the sines and cosines of the sides.

Let O be the centre of the sphere on which the triangle ABC is situated, draw the radii OA, OB, OC; from OA draw the perpendiculars AD and AE, the one in the plane OAB, and the other in the plane OAC, and suppose them to

meet the radii OB and OC produced in D and E. The angle DAE is equal to the angle A of the spherical triangle, and taking the radius unity we have AD = tan C, OD = sec C, AE = tan b, OE = sec b.

Then in triangles DAE and DOE we have

$$OD^2 + OE^2 - 2OD \cdot OE \cos EOD = DE^2$$
$$AD^2 + AE^2 - 2AD \cdot AE \cos A = DE^2$$

by subtracting the second equation from the first, observing that $OD^2 - AD^2 = OE^2 - AE^2 = 1$, and EOD is measured by BC or a, we obtain

$$2 + 2AD \cdot AE \cos A - 2OD \cdot OE \cos a = 0;$$

or by substituting the above values

$$1 + \tan b \cdot \tan c \cos A - \sec b \sec c \cos a = 0$$

$$\text{but } \sec b = \frac{1}{\cos b}, \tan b = \frac{\sin b}{\cos b};$$

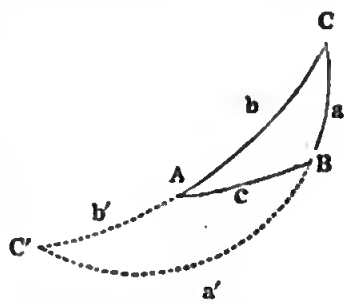
$$\sec c = \frac{1}{\cos c}, \tan c = \frac{\sin c}{\cos c};$$

$$\therefore 1 + \frac{\sin b \sin c \cos A}{\cos b \cos c} - \frac{\cos a}{\cos b \cos c} = 0;$$

hence $\cos a = \cos b \cos c + \sin b \sin c \cos A$ ..... (1)

which is the fundamental formula in Spherical Trigonometry.

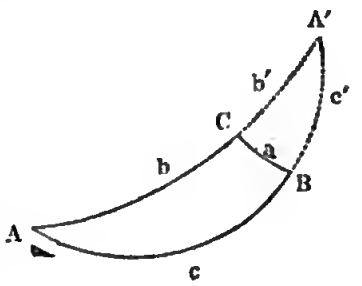
11. In the figure the sides b and c are less than 90°, but it is easily seen that equation (1) is general. Let us suppose that one of the sides, AC or b for example, is greater than 90°; draw the semi-circumferences CAC′, CBC′, and make the triangle ABC′ of which the sides a' and b', or BC′ and AC′, are supplements of a and b, and the angle BAC′ the supplement of A. Since the sides b and c are less than 90°, the equation (1) can be applied to the triangle ABC′, and gives



$$\cos a' = \cos b' \cos c + \sin b' \sin c \cos BAC' \ldots (2).$$

Now $a' = 180° - a$, $b' = 180° - b$, $BAC' = 180° - A$; these values substituted in eq. (2) will give eq. (1), which shows that it is true for the case where b is greater than 90°.

Let us now suppose that the two sides b and c are both greater than 90°; produce AB and AC till they intersect in A′, which forms the triangle BCA′ in which the angle A′ is equal to A, and the sides b' and c' the supplements of b and c; by making the substitutions in this case, we still find that equation (1) satisfied.



Lastly, we can verify equation (1) in the case where $b = 90°$ and $c = 90°$ either both together or separately.

If we apply equation (1) to each of the sides of the triangle, we shall have three equations by means of which we can always find any three parts whatever of the triangle, when the three others are given. But, for practice, it is necessary to have separately the divers relations which exist between four parts of the triangle taken in every possible manner. There are in all four distinct combinations, which we proceed to give.

12. 1st, Relation between the three sides and an angle.

By applying equation (1) to the three angles, we have

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \quad \ldots\ldots \quad (1)$$

$$\cos b = \cos a \cos c + \sin a \sin c \cos B \quad \ldots\ldots \quad (2)$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \quad \ldots\ldots \quad (3)$$

13. 2nd, Relation between two sides and their opposite angles.

From equation (1) we have

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Hence $$\sin^2 A = 1 - \cos^2 A = 1 - \frac{(\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$\frac{\sin A}{\sin a} = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}}{\sin a \sin b \sin c}$$

We must take the radical with the positive sign, seeing that the angles and the sides are less than 180°; their sines are positive. As the second member remains constant when we change A and a into B and b, &c., we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Hence in any spherical triangle, the sines of the angles are to each other as the sines of their opposite sides.

14. 3rd, Relation between the two sides and their included angle, and the angle opposite one of them.

In considering the combination a, b, A, C; first eliminate cos c, between the equations (1) and (3) and we have

$$\cos a = \cos a \cos^2 b + \cos b \sin a \sin b \cos C + \sin b \sin c \cos A$$

transposing $\cos a \cos^2 b$, and observing that $\cos a - \cos a \cos^2 b = \cos a \sin^2 b$; and, dividing the whole by $\sin b \sin a$, it becomes

$$\frac{\cos a \sin b}{\sin a} = \cos b \cos C + \frac{\sin c \cos A}{\sin a},$$

but $\frac{\sin c}{\sin a} = \frac{\sin C}{\sin A}$; and consequently we have for the relation sought

$$\cot a \sin b = \cos b \cos C + \sin C \cot A.$$

By permuting the letters, we have in all the following six equations:

$$\cot a \sin b = \cos b \cos C + \sin C \cot A \ldots\ldots \quad (5)$$
$$\cot b \sin a = \cos a \cos C + \sin C \cot B \ldots\ldots \quad (6)$$
$$\cot a \sin c = \cos c \cos B + \sin B \cot A \ldots\ldots \quad (7)$$
$$\cot c \sin a = \cos a \cos B + \sin B \cot C \ldots\ldots \quad (8)$$
$$\cot b \sin c = \cos c \cos A + \sin A \cot B \ldots\ldots \quad (9)$$
$$\cot c \sin b = \cos b \cos A + \sin A \cot C \ldots\ldots \quad (10)$$

15. 4th, Relation between one of the sides and the three angles. Eliminate b and c from the equations (1) (2) (3): to do this we have by the last article

$$\frac{\cos a \sin b}{\sin a} = \cos b \cos C + \frac{\sin c \cos A}{\sin a},$$

$$\text{and since } \frac{\sin b}{\sin a} = \frac{\sin B}{\sin A} \text{ and } \frac{\sin c}{\sin a} = \frac{\sin C}{\sin A},$$

we have

$$\cos a \sin B = \cos b \sin A \cos C + \cos A \sin C;$$

and changing a and A into b and B, and *vice versâ*, we obtain

$$\cos b \sin A = \cos a \sin B \cos C + \cos B \sin C.$$

We have only now to eliminate cos b by the two preceding equations. We find after reduction the relation sought between A B C and a, which, applied to the three angles successively, will give the three equations

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a \ldots\ldots, (11)$$

$$\cos B = -\cos A \cos C + \sin A \sin C \cos b \ldots\ldots (12)$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c \ldots\ldots (13)$$

16. The analogy of these equations with the fundamental formula is striking, and conducts us to a remarkable consequence. Let us imagine a spherical triangle A′ B′ C′, of which the sides a' b' c' are the supplements of the angles A, B, C; then from equation (1) we shall have

$$\cos a' = \cos b' \cos c' + \sin b' \sin b' \cos A'.$$

Now $\sin a' = \sin A$, $\cos a' = -\cos A$, $\sin b' = \sin B$, &c., then

$$-\cos A = \cos B \cos C + \sin B \sin C \cos A'.$$

From this equation we find for cos A′ a value equal but of a contrary sign to that which we find for cos a in equation (11); then $a = 180° - A'$, similarly $b = 180 - B'$, and $c = 180 - C'$. Hence, having given any spherical triangle, if we describe another triangle, the sides of which are the supplements of the angles of the first, then the sides of the first will be the supplements of the angles of the second. From this property the two triangles are called supplementary, and sometimes the triangles are said to be polar to each other.

NAPIER'S ANALOGIES.

17. We now proceed to deduce the formulæ known by the name of the analogies of Napier, which are employed to simplify some of the cases of spherical triangles.

The equations (1) and (2) give

$$\cos a - \cos b \cos c = \sin b \sin c \cos A;$$
$$\cos b - \cos a \cos c = \sin a \sin c \cos B.$$

By division, observing that $\frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}$,

we have $$\frac{\cos b - \cos a \cos c}{\cos a - \cos b \cos c} = \frac{\sin A \cos B}{\sin B \cos A}.$$

By subtracting and adding unity to both sides of this equation and again dividing

$$\frac{\cos b - \cos a}{\cos b + \cos a} \times \frac{1 + \cos c}{1 - \cos c} = \frac{\sin (A - B)}{\sin (A + B)}.$$

But by Plane Trigonometry, page 30,

$$\frac{\cos b - \cos a}{\cos b + \cos a} = \tan \tfrac{1}{2}(a+b) \tan \tfrac{1}{2}(a-b)$$

$$\text{but } \frac{1 + \cos c}{1 - \cos c} = \frac{1}{\tan^2 \frac{1}{2} c}$$

$$\sin (A + B) = 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A + B)$$
$$\sin (A - B) = 2 \sin \tfrac{1}{2}(A - B) \cos \tfrac{1}{2}(A - B).$$

Substituting these values, the above equation becomes

$$\tan \tfrac{1}{2}(a + b) \tan \tfrac{1}{2}(a - b) =$$
$$\tan^2 \tfrac{1}{2} c \left(\frac{\sin \frac{1}{2}(A - B) \cos \frac{1}{2}(A - B)}{\sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A + B)} \right). \quad (a)$$

$$\text{and since } \frac{\sin a}{\sin b} = \frac{\sin A}{\sin B}$$

$$\text{we have } \frac{\sin a + \sin b}{\sin a - \sin b} = \frac{\sin A + \sin B}{\sin A - \sin B}.$$

By Plane Trigonometry, page 30,

$$\frac{\tan \frac{1}{2}(a + b)}{\tan \frac{1}{2}(a - b)} = \frac{\sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)}{\cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)}.$$

Multiply these two equations together, and then dividing one by the other and extracting the root, observing that $\tan \frac{1}{2}(a+b)$ and $\cos \frac{1}{2}(A+B)$ ought to have the same sign,

$$\tan \tfrac{1}{2}(a+b) = \tan \tfrac{1}{2} c \cdot \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \quad \ldots\ldots\ldots\ldots (14)$$

$$\tan \tfrac{1}{2}(a-b) = \tan \tfrac{1}{2} c \cdot \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \quad \ldots\ldots\ldots\ldots (15)$$

To apply these to the polar triangle we must replace a, b, c, A, B, C, by $180° - A$, $180° - B$, $180° - C$, $180° - a$, $180° - b$, and there results,

$$\tan \tfrac{1}{2}(A+B) = \cot \tfrac{1}{2} C \cdot \frac{\cos \frac{1}{2}(a-b)}{\cos \frac{1}{2}(a+b)} \quad \ldots\ldots\ldots (16)$$

$$\tan \tfrac{1}{2}(A-B) = \cot \tfrac{1}{2} C \cdot \frac{\sin \frac{1}{2}(a-b)}{\sin \frac{1}{2}(a+b)} \quad \ldots\ldots\ldots (17)$$

My able and talented friend, Mr. Reynolds of Chelsea Hospital, has sent me the following very neat method of deducing Napier's Analogies, which he says was communicated to him by Mr. Adams, the celebrated astronomer of Cambridge

$$\text{Let } m = \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin A + \sin B}{\sin a + \sin b}.$$

Then by the formula (11) page 7,

$$\cos A + \cos B \cos C = \sin B \sin C \cos a$$
$$= m \sin C \sin b \cos a \ldots (1)$$
$$\cos B + \cos A \cos C = \sin A \sin C \cos b$$
$$= m \sin C \sin a \cos b \ldots (2)$$

Add (1) and (2), then

$$\{\cos A + \cos B\}(1 + \cos C) = m \sin C \sin (a+b).$$

Also $\sin A + \sin B = m (\sin a + \sin b)$.

Dividing and reducing we have

$$\tan \frac{A+B}{2} \cdot \tan \frac{C}{2} = \frac{\cos \frac{a-b}{2}}{\cos \frac{a+b}{2}} \quad \ldots\ldots\ldots\ldots (\text{Anal. } 1)$$

Again, subtracting (2) from (1)

$$(\cos B - \cos A)(1 - \cos C) = m \cdot \sin(a - b) \sin C$$

and $$\sin B + \sin A = m(\sin a + \sin b);$$

$\therefore$ dividing

$$\tan \frac{A - B}{2} \cdot \tan \frac{C}{2} = \frac{\text{n} \frac{a - b}{2}}{\sin \frac{a + b}{2}} \quad \ldots\ldots\ldots\ldots \text{(Anal. 2)}$$

The other two follow of course from the polar triangle.

ON RIGHT-ANGLED SPHERICAL TRIANGLES.

18. The preceding formulæ will apply to right-angled triangles, if we make any one of the angles $= 90°$.

If $A = 90°$ we have

$\cos a = \cos b \cos c$..... (1)

$\sin b = \sin a \sin B$..... (2) $\quad \sin c = \sin a \sin C$..... (7)

$\tan b = \tan a \cos C$..... (3) $\quad \tan c = \tan a \cos B$..... (8)

$\tan b = \sin c \tan B$..... (4) $\quad \tan c = \sin b \tan C$..... (9)

$\cos B = \sin C \cos b$..... (5) $\quad \cos C = \sin B \cos c$.....(10)

$\cos a = \cot B \cot C$..... (6)

These six independent formulæ are all adapted to logarithmic calculation.

The first gives a relation between the hypothenuse and the two sides containing the right angle; the second, one side and angle opposite; the third, between the hypothenuse, a side, and the adjacent angle; the fourth, between the two sides and the angle opposite to one of them; the fifth, between one side and the two oblique angles; lastly, the sixth, between the hypothenuse and the oblique angles.

19. The formula (1) requires that $\cos a$ must have the same sign as the product $\cos b \cos c$, or that the three cosines must be positive, or that only one must be so. Therefore in any right-angled spherical triangle the three sides must be less than 90°; or two of them must be greater than 90°, and the third less. The formula (4) shows that $\tan b$ has the same sign as $\tan B$, and $\tan c$ the same sign as $\tan C$. Therefore each side containing the right angle is of the same kind or affection as the angle opposite, that is, the angle and the side are both less than 90° or both greater.

NAPIER'S CIRCULAR PARTS.

20. As we have before observed, the above formulæ are simple and well adapted for logarithmic computation, yet they are not easily remembered; therefore it is of importance that we should have some method which will relieve the memory as much as possible; this is supplied by what is termed Napier's Circular Parts. By committing to memory the two rules which will be given hereafter, the student will be able to solve all the cases in right-angled triangles, as well as if he had all the formulæ by heart.

The circular parts of a right-angled spherical triangle are five, namely, the two sides, the complement of the hypothenuse, and the complements of the two angles (the right angle being always omitted).

Three of these circular parts, besides the right angle, enter every proportion, two of which are given, and the third sought.

These three parts are named from their positions with respect to one another, that is, according as they are joined or disjoined, observing that the right angle does not separate the sides.

If the three circular parts join, that which is in the middle is called the *middle part*, and the other two are called *extremes conjunct*.

If the three circular parts do not join, two out of the five must, and that part which is separate or alone is the *middle part*, and the other two are called *extremes disjunct**.

These things being understood, the following is the general rule.

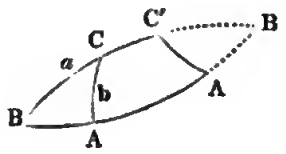
The sine of the middle part is equal to the product of the tangents of the extremes conjunct.

* Thus, if in figure page 12 we suppose B C, the angle B, and the side A B to be the quantities that are to be used; now as they lie all together, the angle B is the middle part, and the two sides, B C and A B, are the extremes conjunct. Also, if the angle B, A B and A C be the quantities, then since the right angle does not separate the sides, A B is the middle part, and the other two elements are the extremes conjunct. But if the quantities be A C, B C and the angle B, then the angle C is said to separate A C from B C, and the side A B is said to separate A C from the angle B, that part A C which is separated from both the others, call the middle part, and the parts which are disjoined from it call extremes disjunct. This practical method will be useful to seamen, and requires very little effort of memory.

The sine of the middle part, is equal to the product of the cosines of the extremes disjunct.

From these two equations, proportions may be formed, observing always to take the complements of the angles and hypothenuse; and that the cosine of a complement is a sine, and the tangent of a complement is a co-tangent, and *vice versâ*.

21. Case 1. When the hypothenuse B C and the base A B are given to find the remaining parts of the triangle.



Let us first proceed to find A C.

Here the hypothenuse and the two sides are the three circular parts.

The hypothenuse being separated or disjoined from the sides it is the middle part, and the sides are the extremes disjunct.

Then sin B C = cos A B cos A C.

And since we must always take the complements of the hypothenuse and angles, this becomes

$$\cos a = \cos b \cos c.$$

Now, as this agrees with equation (1), the rule is proved in this case.

To find the angle B.

Here the three circular parts all lie together, taking B to be the middle part, then A B and B C are adjacent parts, or extremes conjunct.

$$\therefore \sin B = \tan BC \cdot \tan AB;$$

taking the complements of B and B C, we have,

$$\cos B = \cot a \tan c,$$

which corresponds with (8), and therefore the rule is proved in this case also.

To find the angle C.

Here the side A B is separated from the hypothenuse by the angle B, and it is separated from the angle C by the side A C, then A B being the middle part, the hypothenuse and the required angle are the extreme or disjoined parts.

$$\sin AB = \cos BC \cos C;$$

taking the complements of B C and C,

$$\sin c = \sin a \sin C.$$

This agrees with (7), and therefore proves the rule.

22. Case 2. Given the two sides b, and c, which include the right angle, to find the hypothenuse and the angles.

1. To find the hypothenuse.

As the two sides are separated from the hypothenuse they will be extremes disjoined, the hypothenuse being the middle part;

$$\sin BC = \cos AB \cos AC;$$

taking the complement of BC,

$$\cos a = \cos b \cos c; \text{ which is the same as equation (1);}$$

$$\therefore \cos c = \frac{\cos a}{\cos b}.$$

To find angle C.

Since the right angle A does not disjoin, the three parts all lie together, hence AC being the middle part, AB and angle C are the adjacent parts, or extremes conjunct.

$$\sin AC = \tan AB \,.\, \tan C;$$

taking the complement of C,

$$\sin b = \tan c \cot c; \text{ which agrees with (9);}$$

$$\text{or } \tan C = \frac{\tan c}{\sin b}.$$

To find the angle B.

The three circular parts all lie together again, AB being in the middle; calling it the middle part, then AC and angle B will be extremes conjunct.

$$\sin AB = \tan AC \tan B;$$

taking the complement of B,

$$\sin c = \tan b \cot B, \text{ which agrees with (4);}$$

$$\text{or } \tan B = \frac{\tan b}{\sin c}.$$

23. Case 3. Given the hypothenuse a and angle B to find b, c, C.

1. To find AC or b.

As AC is separated from the hypothenuse by the angle C, and from the angle B by the side AB; calling AC the middle part, then BC and angle B are extremes disjunct.

$$\sin AC = \cos BC \cos B;$$

taking the complements of BC and C,

$$\sin b = \sin a \sin B, \text{ which is the same as equation (2).}$$

2. To find AB or c.

Here the three circular parts all lie together, and AB in the middle; calling it the middle part, then AB and BC will be adjacent parts, or extremes conjunct.

$$\sin B = \tan AB \tan BC;$$

taking the complements of B and BC,

$$\cos B = \tan c \cot a;$$

$$\therefore \tan c = \frac{\cos B}{\cot a} = \cos B \tan a, \text{ which agrees with equ. (8).}$$

3. To find C.

Here the circular parts lie all together, the hypothenuse being in the middle; call it the middle part, and the angles B and C will be adjacent parts.

$$\sin BC = \tan B \tan C;$$

taking the complements throughout,

$$\cos a = \cot B \cot C;$$

$$\therefore \cot C = \frac{\cos a}{\cot B} = \cos a \tan B, \text{ which agrees with equ. (6).}$$

24. Case 4. Given the side AC or b and the opposite angle B to find a, c, C.

1. To find the hypothenuse BC or a.

Here b or AC is separated from the hypothenuse by the angle C, and from the angle B by the side AB; calling then AC the middle part, the angle B and the hypothenuse are the extremes disjunct.

$$\sin AC = \cos BC \cos B;$$

taking the complements of BC and B,

$$\sin b = \sin a \sin B \text{ which agrees with equation (2).}$$

$$\sin a = \frac{\sin b}{\sin B}.$$

2. To find c.

As the right angle does not disjoin, c lies in the middle between b and B; calling it the middle part, b and B are the extremes conjunct.

$$\sin C = \tan b \tan B;$$

taking the complement of B,

$$\sin c = \tan b \cot B, \text{ which agrees with equation (4).}$$

3. To find C.

Here the angle B is separated from c by the hypothenuse, and it is separated from b by the side AB; calling it the middle part,

$$\sin B = \cos b \cos C;$$

taking the complements of B and C,

$$\cos B = \cos b \sin C, \text{ which agrees with equation (5).}$$

$$\sin C = \frac{\cos B}{\cos b}.$$

There is here an ambiguity, since each quantity is determined by its sine, and we see that this really ought to be the case. In fact, if the triangle BAC (fig. p. 12) right-angled at A, satisfy the equation; produce BA and BC till they intersect in D, then take $DA' = BA$, and $DC' = BC$, the triangles BAC, $DA'C'$ will be equal in all respects, then the angle A is a right angle, and $C'A' = CA = b$. Thus the triangle $BA'C'$ is right angled, and contains also the given parts B and b; we can therefore take at will $a < 90°$, or $A > 90°$, but when the choice is once made the affection or species of c will be determined by the equation $\cos a = \cos b \cos c$, and that affection will be the same as that of C. There will be only one triangle which has two right angles when $b = B$, and none when we have $\sin b > \sin b$.

25. Case 5. Given the side b and the adjacent angle C, to find a, c, B.

1. To find the hypothenuse a.

Here the parts all lie together, the angle C being in the middle; call it the middle part, then AC and BC are the extremes conjunct.

$$\sin C = \tan AC \tan BC;$$

taking the complements of C and BC,

$$\cos C = \tan b \cot a; \text{ which agrees with equation (3);}$$

$$\therefore \tan a = \frac{\tan b}{\cos C}.$$

2. To find AB or c.

As the right angle does not disconnect, AC is the middle part, and AB and angle C are the extremes conjunct.

$$\sin AC = \tan AB \tan C;$$

taking the complement of C,

$$\sin b = \tan c \cot c;$$

$$\therefore \tan c = \frac{\sin b}{\cot C} = \sin b \tan C, \text{ which agrees with equ. (9).}$$

3. To find the angle B.

Since the angle B is separated from b by the side AB, and from the angle C by the hypothenuse BC; calling it the middle part, then AC and the angle C are the extremes disjunct.

$$\sin B = \cos AC \cos C;$$

taking the complements of B and C,

$$\cos B = \cos b \sin C, \text{ which agrees with equation (5).}$$

Here a, c and B are found without any ambiguity.

26. Case 6. Given the two oblique angles B and C to find a, b, c.

1. To find a or BC.

Here a, B and C all lie together, a or BC being in the middle; call it the middle part, then B and C are the extremes conjunct.

$$\sin BC = \tan B \tan C;$$

taking the complements of the whole,

$$\cos a = \cot B \cot C, \text{ which agrees with equation (6).}$$

2. To find b or AC.

Here B is separated from C by the hypothenuse BC, and it is separated from AC by the side AB; calling it the middle part, then the angle C and AC are the extremes disjunct.

$$\sin B = \cos C \cos AC;$$

taking the complements of B and C,

$$\cos B = \sin C \cos b, \text{ which agrees with equation (5).}$$

3. To find c or AB.

Here the angle C is separated from AB by the side AC, and from the angle B by the hypothenuse BC; calling it the middle part, then AB and the angle B are the extremes disjunct.

$$\sin C = \cos AB \cos B;$$

taking the complements of C and B,

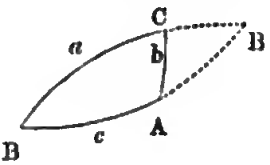
$$\cos C = \cos c \sin B, \text{ which agrees with equation (10);}$$

$$\therefore \cos c = \frac{\cos C}{\sin B}.$$

These values leave no ambignity, and if the triangle is impossible they will show that it is so.

27. When a triangle is isosceles, the two equal sides are only counted as one element, and the angles which are opposite to them also as only one element. Now, if we draw the arc of a great circle through the vertex of the triangle and the middle of the base, we divide it into two right-angled triangles, equal in all respects, and in each of which we know two elements besides the right angle, then the isosceles triangle can be solved by the formulæ for right-angled triangles.

28. If in a spherical triangle ABC, in which we have $a+b=180°$, produce a and c till they intersect in D, we shall have $a+CD=180°$, hence $CD=b$; therefore, the solution of the triangle ABC is brought to that of the isosceles triangle ABC.



The same thing may be said of a triangle, in which two angles are the spherical supplements of each other, for we cannot have $A+B=180°$ without at the same time having $A+B=180°$ and *vice versâ*. In fact, in the isosceles triangle ACD, the angle $CAD=D=B$. Now, $CAD+CAB=180°$; then also, in the triangle ABC we ought to have $A+B=180°$.

CHAPTER II.

SOLUTION OF OBLIQUE ANGLED SPHERICAL TRIANGLES.

29. Case 1. Given the three sides a, b, c to find the angles A, B, C.

To find A, we have by equation (1)

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

but we obtain an expression better adapted to logarithms by finding $\sin \frac{1}{2} A$, $\cos \frac{1}{2} A$, &c., as in Plane Trigonometry.

Since $2 \sin^2 \frac{1}{2} A = 1 - \cos A$, we have by substituting the above value of $\cos A$,

$$2 \sin^2 \tfrac{1}{2} A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c}$$

$$= \frac{\cos (b - c) - \cos a}{\sin b \sin c} =$$

(by equation (8) page 30, Plane Trigonometry,)

$$\frac{2 \sin \frac{1}{2} (a + b - c) \sin \frac{1}{2} (a - b + c)}{\sin b \sin c};$$

$$\therefore \sin \tfrac{1}{2} A = \sqrt{\frac{\sin \frac{1}{2} (a + b - c) \sin \frac{1}{2} (a - b + c)}{\sin b \sin c}}.$$

For the sake of abridgment, put $a + b + c = 2s$, and the preceding expression becomes

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin b \sin c}}$$

In the same way

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin s \sin (s - a)}{\sin b \sin c}};$$

$$\therefore \tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s - b) \sin (s - c)}{\sin s \sin (s - a)}}.$$

30. Case 2, Given the two sides a, b, and the angle A opposite to one of them, to find c, B, C.

We obtain at first the angle B opposite to b by the proportion

$$\sin a : \sin b :: \sin A : \sin B;$$

$$\therefore \sin B = \frac{\sin A \sin b}{\sin a}.$$

It will be best to determine c and C by Napier's Analogies, which give

$$\tan \tfrac{1}{2} c = \tan \tfrac{1}{2} (a - b) \cdot \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)}$$

$$\cot \tfrac{1}{2} C = \tan \tfrac{1}{2} (A - B) \cdot \frac{\sin \frac{1}{2}(a + b)}{\sin \frac{1}{2}(a - b)}.$$

As the angle B is determined by its sine, it can either be acute or obtuse. However, for certain values of the given quantities a, b, A, there will be only one triangle. We may refer back to the similar case of plane triangles, we can thus find C in a direct manner by the equation

$$\cot A \sin C + \cos b \cos C = \cot a \sin b.$$

To effect this, let us at first determine an auxiliary angle φ, by putting $\cot A = \cos b \cot \varphi$, from whence we have

$$\cot \varphi = \frac{\cot A}{\cos b};$$

then in the equation (5), p. 6, $\cot A = \cos b \cot \varphi = \dfrac{\cos b \cos \varphi}{\sin \varphi}$, the equation becomes

$$\cos b (\sin C \cos \varphi + \cos C \sin \varphi) = \cot a \sin b \sin \varphi,$$

from which we find

$$\sin (C + \varphi) = \frac{\tan b \sin \varphi}{\tan a};$$

hence $C + \varphi$ is determined; let $C + \varphi = m$, and we have $C = m - \varphi$.

After having found C, we obtain the side c by the proportion

$$\sin A : \sin C :: \sin a : \sin c.$$

But if we wish to find c directly, we must refer back to equation (1), page 5,

$$\cos b \cos c + \cos A \sin b \sin c = \cos a.$$

This may be reduced in the same way as the equation above, by using an auxiliary angle φ, putting $\cos A \sin b = \cos b \cot \varphi$, from whence we have

$$\cot \varphi = \cos A \tan b;$$

consequently, the above equation becomes

$$\cos b (\sin \varphi \cos c + \cos \varphi \sin c) = \cos a \sin \varphi, \text{ or}$$

$$\sin (c + \varphi) = \frac{\cos a \sin \varphi}{\cos b}.$$

Having found φ, we can easily find c.

31. Case 3. Given the two sides a and b and the included angle C to find A, B, c.

The formulæ (5) (6), page 6, give for A and B

$$\cot A = \frac{\cot a \sin b - \cos b \cos C}{\sin C}$$

$$\cot B = \frac{\cot b \sin a - \cos a \cos C}{\sin C}$$

By employing auxiliary angles it is easy to reduce each numerator to a single quantity, but is is more simple to recur to Napier's Analogies.

$$\tan \tfrac{1}{2}(A + B) = \cot \tfrac{1}{2} C \cdot \frac{\cos \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2}(a + b)}$$

$$\tan \tfrac{1}{2}(A - B) = \cot \tfrac{1}{2} C \cdot \frac{\sin \tfrac{1}{2}(a - b)}{\sin \tfrac{1}{2}(a + b)}$$

which give $\frac{1}{2}(A + B)$ and $\frac{1}{2}(A - B)$, consequently by adding and subtracting we find A and B.

The angles being found we obtain c from the proportion $\sin A : \sin C :: \sin a : \sin c$; but if we wish to have c directly we must take the formula, page 5,

$$\cos c = \cos a \cos b + \sin a \sin b \cos C;$$

in which if we make $\sin b \cos C = \dfrac{\cos b \cos \varphi}{\sin \varphi}$

$= \cos b \cot \varphi$, then it becomes without any ambiguity

$$\cot \varphi = \tan b \cos C, \ \therefore \cos c = \frac{\cos b \sin (a + \varphi)}{\sin \varphi}.$$

32. Case 4. Given the two angles A and B, and the adjacent side c, to find a, b, c.

We can find a and b by the formulæ (7) and (9), page 6,

$$\cot a = \frac{\cot A \sin B + \cos B \cos c}{\sin c}$$

$$\cot b = \frac{\cos B \sin A + \cos A \cos c}{\sin c}$$

and better still by Napier's Analogies,

$$\tan \tfrac{1}{2}(a + b) = \tan \tfrac{1}{2} c \cdot \frac{\cos \tfrac{1}{2}(A - B)}{\cos \tfrac{1}{2}(A + B)}$$

$$\tan \tfrac{1}{2}(a - b) = \tan \tfrac{1}{2} c \cdot \frac{\sin \tfrac{1}{2}(A - B)}{\sin \tfrac{1}{2}(A + B)}.$$

These equations determine $\frac{1}{2}(a + b)$ and $\frac{1}{2}(a - b)$, and from which, by adding and subtracting, we find a and b.

We can now find C by the proportion

$$\sin a : \sin c :: \sin A : \sin C,$$

or we can find C directly by making use of the formula, equation (13), page 7, viz.,

$$\cos C = \sin A \sin B \cos c - \cos A \cos B.$$

If we put $\sin B \cos c = \cos B \cot \varphi$, it will become

$$\cot \varphi = \tan B \cos c, \cos C = \frac{\cos B \sin (A - \varphi)}{\sin \varphi}.$$

This case is analogous to the third case, and offers no ambiguity.

33. Case 5. Given the two angles A and B, and the side a opposite to one of them, to find b, c, C.

This case is quite analogous to the second, and is treated in the same manner, and has the same ambiguities.

We deduce b from the proportion

$$\sin A : \sin B :: \sin a : \sin b,$$

and we find c and C by the formulæ already employed,

$$\tan \tfrac{1}{2} c = \tan \tfrac{1}{2}(a - b) \cdot \frac{\sin \tfrac{1}{2}(A + B)}{\sin \tfrac{1}{2}(A - B)},$$

$$\cot \tfrac{1}{2} C = \tan \tfrac{1}{2}(A - B) \cdot \frac{\sin \tfrac{1}{2}(a + b)}{\sin \tfrac{1}{2}(a - b)}.$$

The side c can also be obtained by equation (7),

$$\cot a \sin c - \cos B \cos c = \cot A \sin B,$$

in which we make $\cot a = \cos B \cot \varphi$

$$\therefore \cot \varphi = \frac{\cot a}{\cos B}, \quad \sin (c - \varphi) = \frac{\tan B \sin \varphi}{\tan A}.$$

Lastly, we can find C, for $\sin a : \sin c :: \sin A : \sin C$, or better by means of the equation,

$$\cos a \sin B \sin C - \cos B \cos C = \cos A,$$

we reduce the first member to a monomial by putting $\cos a \sin B = \cos B \cot \varphi$, from whence we have

$$\cot \varphi = \cos a \tan B, \quad \sin (C - \varphi) = \frac{\cos A \sin \varphi}{\cos B};$$

these values determine φ, $C - \varphi$, and consequently the angle C.

34. Case 6. Given the three angles A, B, C, to find the sides a, b, c.

This case is solved in a similar way to the first.

By page 7, equation (11)

$$\cos a = \frac{\cos a + \cos B \cos C}{\sin B \sin C},$$

and by the same method, as used in the first case,

$$\sin \tfrac{1}{2} a = \sqrt{\frac{\sin S \sin (A - S)}{\sin B \sin C}}$$

$$\cos \tfrac{1}{2} a = \sqrt{\frac{\sin (B - S) \cdot \sin (C - S)}{\sin B \sin C}}$$

$$\tan \tfrac{1}{2} a = \sqrt{\frac{\sin S \sin (A - S)}{\sin (B - S) \sin (C - S)}}.$$

By using the polar triangle in Case 1, we have

$$\sin \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cdot \cos (S - A)}{\sin B \sin C}}$$

$$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}}$$

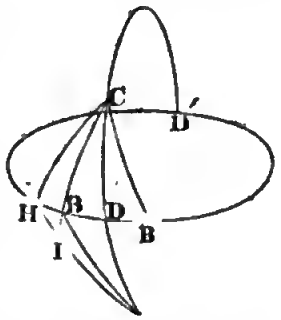
$$\tan \tfrac{1}{2} a = \sqrt{\frac{-\cos S \cos (S - A)}{\cos (S - B) \cos (S - C)}}.$$

The first and last of these appear under an impossible form, but since S is always greater than 90 and less than 270, the cos S is always negative, and therefore makes the quantity under the radical always positive.

ON THE AMBIGUOUS CASES OF SPHERICAL TRIANGLES.

35. The only cases in which there is any uncertainty are the second and fifth. We proceed to show in this article what conditions are necessary that there may be two solutions, or only one, or even when the triangle is impossible.

Let us consider upon a sphere a semicircle DCD′ perpendicular to a whole circle DHD′; take CD less than 90°, and draw the arcs of great circles CB, CB′, CH . . . . from the point C to the different points of the circumference DHD′. Produce CD, making C′D = CD, and join C′B. The triangles CDB, C′DB have a right angle contained between the equal sides, therefore CB = C′B. Now we have CDC′ < CB + BC′, therefore CD < CB.



Hence, in the first place, the arc CD is the least that we can draw from the point C to the circumference DHD′; and consequently CD′ is the greatest.

Let DB′ = DB; then in the two triangles CDB and CDB′ have the two sides CD, CB and the right angle CDB of the one, equal to the two sides CD, DB, and the right angle CDB′ of the other, hence CB′ = CB. Therefore, in the second place, the oblique arcs equally distant from CD or CD′ are equal.

Lastly, let DH > DB; draw C′H and produce CB till it intersects C′H in I. Then, since the arc CC′ is less than a semicircle, it will meet CB produced beyond the point C′; this requires that the intersection I falls between H and C′. We have therefore C′B < C′I + IB, and consequently C′B + BC < C′I + IC. But we have IC < IH + HC, and therefore C′I + IC < C′H + HC; hence, *à fortiori*, C′B + BC < C′H + HC. Now, C′B = BC and C′H = HC, therefore we have BC < HC. Consequently, in the third place, the oblique arcs are greater the farther they are from CD, or the more they approach CD′.

Now, suppose we have to construct a spherical triangle, the given quantities being *a*, *b*, and the angle A opposite to *a*.

We may at first remark that certain cases of impossibility are indicated even by the calculation. To show this, make

the angle CAB = A and AC = b, produce AC and AB till they intersect in E, then let fall the perpendicular CD upon AE.

The arc CD ought to be of the same affection or species as

the angle A by Art. 19; then, when A is acute, CD is the shortest distance from the point C to the semi-circumference AE, and it is the greatest when A is obtuse.

In the first hypothesis the triangle will be impossible if we have $a <$ CD, which gives $\sin a <$ sin CD; and in the second it will be impossible if we have $a >$ CD, which gives again sin A > sin CD.

Now, in the right-angled spherical triangle ACD, we have

$$\sin \text{CD} = \sin b \sin \text{A};$$

then, in both hypotheses we shall have $\sin a < \sin b \sin$ A. On the other hand, when we seek the angle B of the unknown triangle ACB, we have

$$\sin \text{B} = \frac{\sin b \sin \text{A}}{\sin a};$$

then this value of sin B will be > 1, which is impossible.

If we have $a =$ CD, there will be only one right-angled triangle, ACD, which will be possible, and it is that which again indicates the value of sin B, which becomes sin B = 1. It is understood that the angle A is not equal to 90°.

Let us now examine the different relations of magnitude which the given quantities a, b, A can present.

Let A < 90° and $b <$ 90° (fig. p. 23). Since A and b are < 90°, AD is also < 90° by Art. 19; then AD < DE; if now we have besides $a < b$, it is clear that we can place between CA and CD an arc CB = a, and that on the other side, between CD and CE, we can put another CB′ = CB = a; that is to say, there are two triangles ACB and ACB′ which have the same quantities given, viz., a b A.

When $a = b$, the triangle ACB disappears, and there remains only the triangle ACB′.

When $a + b = 180$, or when $a + b > 180$, the point B′ coincides with E, or passes beyond it, and then no triangle can exist.

We can discuss in the same manner the other hypotheses. The results are all contained in the following table. The sign $\geqslant$ signifies equal to or greater than; and the sign $\leqslant$ signifies equal to or less than.

A < 90°	b < 90°	$a < b$	two solutions.
		$a \geqslant b$	one solution.
		$a + b \geqslant 180°$	no solution.
	b > 90°	$a + b < 180°$	two solutions.
		$a + b \geqslant 180°$	one solution.
		$a \geqslant b$	no solution.
	b = 90°	$a < b$	two solutions.
		$a \geqslant b$	no solution.
A > 90°	b < 90°	$a + b > 180°$	two solutions.
		$a + b \leqslant 180°$	one solution.
		$a \leqslant b$	no solution.
	b > 90°	$a > b$	two solutions.
		$a \leqslant b$	one solution.
		$a + b \leqslant 180°$	no solution.
	b = 90°	$a > b$	two solutions.
		$a \leqslant b$	no solution.
A = 90°	b < 90°	$a > b$	one solution.
		$a \leqslant b$	no solution.
		$a + b \geqslant b$	no solution.
	b > 90°	$a < b$	one solution.
		$a \geqslant b$	no solution.
		$a + b \leqslant 180°$	no solution.
	b = 90°	$a = 90°$	solutions ad infinitum.
		$a <$ or $> 90°$	no solution.

By the properties of the polar triangle, we can apply the results to the fifth case, where A, B, *a*, are given, only taking care to change *a*, *b*, A into A, B, *a*, the sign $>$ into $<$, and the sign $<$ into $>$.

When the given quantities fall in a case where we ought to have only one solution, the calculation will still indicate two. But to discern which ought to be taken, it is sufficient to observe, that the greater angle must be opposite to the greater side, and conversely.

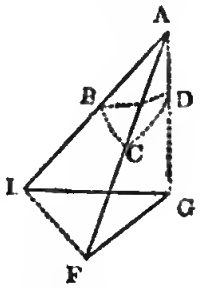
See *Lefebure De Fourcy's Trigonometry.*

Suppose, for example, that we have given A=112°, a=102°, b = 106°. In the preceding table, among the cases which correspond to A > 90°, we consider that where $b > 90°$, and among these that where $a \lessdot b$. We may observe besides, that $a + b = 208°$, therefore $a + b > 180°$, we conclude from the table that there is only one solution, and since b is > a, the angle B is greater than A, therefore B is obtuse.

TO REDUCE AN ANGLE TO THE HORIZON.

36. Let BAC be an angle in an inclined plane, and AD the vertical passing through A. Draw the horizontal plane meeting the lines AB, AC, AD, in E, F, G; the angle EGF is the horizontal projection of the angle BAC, or, in other words, it is the angle BAC reduced to the horizon. It is this angle EFG that we have to calculate, supposing the angles BAC, BAD, CAD, to have been determined by an instrument.

The geometrical construction is easy, for the line AG being arbitrary, we shall have sufficient quantities given to construct at first the right-angled triangle, EAG and FAG, then the triangle EAF, aud, lastly, the triangle EGF. The calculation of the angle EGF is equally easy. If we describe a sphere from the centre A with any radius, the lines AB, AC, AD, where they meet the sphere, will determine a spherical triangle BCD, of which the sides are known by means of the given angles, and of which the angle BDC of the triangle is equal to the required angle EGF.



Then by the first case of oblique-angled spherical triangles, page 18, we have

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}};$$

where a = BAC; b = BAD; c = CAD; $s = \frac{1}{2}(a+b+c)$.

Let a=47° 45′ 39″, b=69° 49′ 19″, c=80° 17′ 36″. We shall have $2s$=197° 52′ 34″, s=98° 56′ 17″; $s-b$=29° 6′ 58″; $s-c$=18° 38′ 41″.

log sin $(s-b)$	.....	9·6871552
log sin $(s-c)$	.....	9·5047412
comp. log sin b	.....	0·0275078
comp. log sin c	.....	0·0062623
2 log sin $\frac{1}{2}$ A		19·2256665
log sin $\frac{1}{2}$ A		9·6128332

$\therefore \frac{1}{2}$ A = 24° 12′ 27″·9, or A = 48° 24′ 56″.

37. The following properties of spherical triangles we shall premise before entering on the numerical solution of triangles.

Any side of a spherical triangle is less than a semicircle, and any angle is less than two right angles.

For the limit of any plane angle is two right angles, and this is also the limit of any plane face of a solid angle.

The sum of the three angles is greater than two right angles and less than six right angles.

If the three sides of a spherical triangle be equal, the three angles will also be equal, and *vice versâ*.

If the sum of any two sides of a spherical triangle be equal to 180°, the sum of their opposite angles will also be equal to 180°, and *vice versâ*.

If the three angles of a spherical triangle be all acute, all right, or all obtuse, the three sides will be accordingly all less than 90°, all equal to 90°, or all greater than 90°, and *vice versâ*.

The sum of any two sides is greater than the third side, and their difference is less than the third side.

The sum of any two angles is greater than the supplement of the third angle.

The sum of the three sides is less than the circumference of a great circle.

If any two sides of a triangle be equal to each other, their opposite angles will be equal, and *vice versâ*.

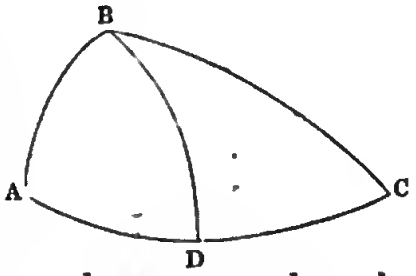
Since

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

$$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c}.$$

If $b = a$, these expressions are each $= \dfrac{\cos a - \cos a \cos c}{\sin a \sin c}$

$\therefore \cos A = \cos B$ or $A = B$;



that is, the angles at the base of an isosceles triangle are equal, and the converse of this holds also. From this it is easily shown that the greater side of a spherical triangle is opposite the greater angle, for let A B C be greater than C A B, and make the angle A B D equal to the angle D A B; $\therefore$ D A = D B.

$$A C = A D + D C = D C + D B, \text{ but } D C + D B > B C$$
$$\therefore A C > B C.$$

ON THE NUMERICAL SOLUTION OF RIGHT-ANGLED SPHERICAL TRIANGLES.

38. When the hypothenuse and one side are given.

Ex. 1. Given the hypothenuse B C = 63° 56′ 7″, and the side A B = 40°, to find the remaining parts of the triangle.

To find the other side, A C.

Here the hypothenuse and the two sides are the three circular parts.

The hypothenuse being separated or disjoined from the sides by the angles is therefore the middle part, and the sides the extremes disjunct.

$$\sin B C = \cos A B \cos A C;$$

taking the complement of hypothenuse as directed by the rule,

$$\cos B C = \cos A B \cos A C;$$

$$\log \cos B C = \log \cos A B + \log \cos A C - 10$$
$$\log \cos A C = \log \cos B C - \log \cos A B + 10$$
$$= \log \cos 63° 56′ 7″ - \log \cos 40° + 10$$
$$= 9{\cdot}6428464 - 9{\cdot}8842540 + 10$$
$$= 9{\cdot}7585924;$$
$$\therefore A C = 54° 59′ 59″{\cdot}6.$$

The side A C is acute, because the hypothenuse and the given side have the same affection.

To find the angle B.

This angle connects the hypothenuse and the given side, and is therefore the middle part, and the other the extremes conjunct.

$$\therefore \sin B = \tan AB \,.\, \tan BC;$$

taking the complements of the angle and hypothenuse,

$$\cos B = \tan AB \,.\, \cot BC;$$

$$\begin{aligned}\log \cos B &= \log \tan AB + \log \cot BC - 10\\ &= \log \tan 40^\circ + \log \cot 63^\circ\ 56'\ 7''\\ &= 9{\cdot}9238135 + 9{\cdot}6894258 - 10\\ &= 9{\cdot}6132393;\end{aligned}$$

$$\therefore B = 65^\circ\ 46'\ 5''.$$

The angle B is acute, as the hypothenuse and given side are of the same affection.

To find the angle C.

Here the side A B is separated from the hypothenuse by the angle B, and it is separated from the angle C by the side A C; take it to be the middle part, then B C and the angle A are extremes disjunct.

$$\sin AB = \cos BC \,.\, \cos C;$$

taking the complements of hypothenuse and angle C,

$$\sin AB = \sin BC \sin C;$$

$$\log \sin AB = \log \sin BC + \log \sin C - 10,$$

$$\begin{aligned}\log \sin C &= \log \sin AB - \log \sin BC + 10\\ &= \log \sin 40^\circ - \log \sin 63^\circ\ 56'\ 7'' + 10\\ &= 9{\cdot}8080675 + 0{\cdot}0465794 \text{ by taking comp. log. } 63^\circ\ 56'\ 7'';\\ &= 9{\cdot}8546469;\end{aligned}$$

$$\therefore C = 45^\circ\ 41'\ 21''.$$

The angle C is acute, the hypothenuse and given side being of the same affection.

When the two sides are given.

Given the side A C = 52° 13′, and the side A B = 42° 17′, to find the remaining parts.

To find the angle B. (See fig. p. 12.)

As the right angle does not disjoin, A B is the middle part, and the angle and side A C are extremes conjunct.

$$\sin AB = \tan B \,.\, \tan AC;$$

taking the complement of B,

$$\sin AB = \cot B \tan AC;$$

$$\begin{aligned}\log \cot B &= \log \sin AB - \log \tan AC + 10\\ &= 9{\cdot}8278843 + 10 - 10{\cdot}1105786\\ &= 9{\cdot}7173057, \text{ which is the log cot } 62° 27′;\end{aligned}$$

$$\therefore B = 62° 27′,$$

which is acute, like its opposite side.

To find the angle C.

Here A C is the middle part, and the angle C and A B are extremes conjunct.

$$\sin AC = \tan AB \tan C;$$

taking the complement of C

$$\sin AC = \tan AB \cot C;$$

$$\begin{aligned}\log \cot C &= \log \sin AC - \log \tan AB + 10\\ &= 9{\cdot}8978103 - 9{\cdot}9587542 + 10\\ &= 9{\cdot}9390561, \text{ which is the log cot of } 49°.\end{aligned}$$

The angle is acute like its opposite side.

To find the hypothenuse B C.

The hypothenuse being separated from the sides by the angles, it is the middle part, and the sides are the extremes disjunct.

$$\sin BC = \cos AB \,.\, \cos AC;$$

taking the complement of the hypothenuse,

$$\cos BC = \cos AB \cos AC;$$

$$\log \cos BC = \log \cos AB + \log \cos AC - 10$$
$$= 9{\cdot}8691301 + 9{\cdot}7872317 - 10$$
$$= 9{\cdot}6563613, \text{ which is the cosine } 62^\circ\ 31',$$

which is less than 90°, AC and BC being alike.

When a side and its opposite angle are given.

Given the side $AC = 55^\circ$, and its opposite angle $B = 65^\circ\ 46'\ 5''$, to find the remaining parts of the triangle.

To find the other angle C.

Here B is the middle part, being separated from AC by AB, and from the angle C by BC;

$\therefore$ AC and C are the extremes disjunct.

$$\sin B = \cos AC \cos C;$$

taking the complements of B and C,

$$\cos B = \cos AC \sin C;$$
$$\log \sin C = \log \cos B - \log \cos AC + 10$$
$$= 9{\cdot}6132407 + \text{comp. log } 0{\cdot}2414087 + 10$$
$$= 9{\cdot}8546494 = \log \sin 45^\circ\ 41'\ 21'';$$
$$\therefore C = 45^\circ\ 41'\ 21''.$$

The angle C is ambiguous; as it cannot be determined by the data alone whether, AB, C, and BC are greater or less than 90°.

To find the side AB.

Here AB is the middle part, AC and B the extremes conjunct.

$$\sin AB = \tan AC \tan B;$$

taking the complement of B,

$$\sin AB = \tan AC \cot B;$$
$$\log \sin AB = \log \tan AC + \log \cot B - 10$$
$$= 10{\cdot}1547732 + 2{\cdot}6532976 - 10$$
$$= 9{\cdot}8080708, \text{ which is the sin } 40^\circ;$$
$$\therefore AB = 40^\circ.$$

The side AB is also ambiguous for the same reason as above.

To find the hypothenuse BC.

The side AC is the middle part, and BC and B are the extremes disjunct.

$$\sin AC = \cos BC \cdot \cos B;$$

taking the complements of hypothenuse and angle, B,

$$\sin AC = \sin BC \sin B;$$

$$\log \sin AC = \log \sin BC + \log \sin B - 10$$

$$\log \sin BC = \log \sin AC - \log \sin B + 10$$

$$= 9{\cdot}9133645 + 0{\cdot}0400568 + 10$$

$$= 9{\cdot}9534213;$$

$$\therefore BC = 63^\circ\ 56'\ 7''.$$

When a side and its adjacent angle are given.

Given the side AC = 54′ 46′, and its adjacent angle 47°56′, to find the remaining parts.

To find the side AB.

Here the circular parts all lie together, hence AC is the middle part, and AB and C the extremes conjunct.

$$\sin AC = \tan AB \tan C;$$

taking the complement of C.

$$\sin AC = \tan AB \cot C;$$

$$\log AC = \log \tan AB + \log \cot C - 10$$

$$\log \tan AB = \log \sin AC - \log \cot C + 10$$

$$= 9{\cdot}9121207 - 9{\cdot}9554535 + 10$$

$$= 9{\cdot}9566672 \text{ which is the tangent of } 42^\circ\ 8'\ 46'';$$

$$\therefore AB = 42^\circ\ 8'\ 46''$$

which is acute, like its opposite angle.

To find the angle B.

Here B is separated from the two given quantities; calling it the middle part, then AC and C are the extremes disjunct.

$$\sin B = \cos AC \cos C;$$

taking the complements of B and C,

$$\cos B = \cos AC \sin C;$$

$$\begin{aligned}\log \cos B &= \log \cos AC + \log \sin \sin C - 10\\ &= 9{\cdot}7611063 + 9{\cdot}8706179 - 10\\ &= 9{\cdot}6317242, \text{ which is } \cos 64^\circ\ 38'\ 31'';\end{aligned}$$

$$\therefore B = 64^\circ\ 38'\ 31''.$$

To find the hypothenuse BC.

Here the circular parts all lie together, and C being in the middle, is the middle part, and BC and AC the extremes disjunct.

$$\sin C = \tan BC \tan AC;$$

taking the complements of the hypothenuse and of angle C,

$$\cos C = \tan AC \cot BC;$$

$$\log \cos C = \log \tan AC + \log \cot BC - 10;$$

$$\begin{aligned}\therefore \log \cot BC &= \log \cos C - \log \tan AC + 10\\ &= 9{\cdot}8260715 - 10{\cdot}1510145 + 10\\ &= 9{\cdot}6750570, \text{ which is the cotangent of}\\ &\quad 64^\circ\ 40'\ 34'';\end{aligned}$$

$$\therefore BC = 64^\circ\ 40'\ 34''.$$

QUADRANTAL TRIANGLES.

39. Quadrantal triangles can be solved by the same rules as right-angled triangles for using the polar triangle; we see that since one side is a quadrant, and that in the polar triangle $A' = 180 - a$;

$$\therefore A' = 180^\circ - 90^\circ = 90.$$

In the polar triangle, since $A' = 90$, we have by the equations, page 10,

$$\cos a' = \cos b' \cos c'$$

$$\sin b' = \sin a' \sin B' \qquad \sin c' = \sin a' \sin C'$$

$$\tan b' = \tan a' \cos C' \qquad \tan c' = \tan a' \cos B'$$

$\tan b' = \sin c' \tan B'$ $\qquad$ $\tan c' = \sin b' \tan C'$

$\cos B' = \sin C' \cos b'$ $\qquad$ $\cos C' = \sin B' \cos c'$

$\cos a' = \cot B' \cot c'.$

From these by substituting these values

$$a' = 180° - A;\ b' = 180° - B;\ c' = 180° - C;$$
$$A' = 180° - a;\ B' = 180° - b;\ C' = 180° - c;$$

we get these results,

$\cos A = -\cos B \cos C$

$\sin B = \sin A \sin b$ $\qquad$ $\sin C = \sin A \sin b$

$\tan B = -\tan A \cos c$ $\qquad$ $\tan C = -\tan A \cos b$

$\tan B = \tan b \sin c$ $\qquad$ $\tan C = \sin B \tan c$

$\cos b = \sin c \cos B$ $\qquad$ $\cos c = \sin b \cos C$

$\cos A = \cot b \cot c$

Or without using the polar triangle,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \text{ make } a = \text{quadrant},$$

then $\cos a = 0$, and we have

$$\cos A = -\frac{\cos b \cos c}{\sin b \sin c} = -\cot b \cot c;$$

$$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c} = \frac{\cos b}{\sin a};$$

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b} = \frac{\cos c}{\sin b}.$$

From these equations, and the equation $\frac{\sin A}{\sin B} = \frac{\sin a}{\sin b}$, we can deduce all the cases of quadrantal triangles.

Given $A B = c = 32° 57' 6''$ and $A C = b = 66° 32'$, to find B and A,

$$\cos A = -\cot b \cot c$$
$$\log \cos A = \log \cot b + \log \cot c - 10$$
$$= 10{\cdot}1882850 + 9{\cdot}6376106 - 10$$
$$= 9{\cdot}8258956, \text{ which is the cosine of } 47° 57' 16'',$$

but since cos A is negative, A must be greater than 90°.

OBLIQUE-ANGLED TRIANGLES.

40. Case 1. Given the three sides, viz.

AB	=	79° 17′ 14″	} to find the rest.
BC	=	110°	
AC	=	58°	

To find the angle A.

By page 18, $\sin \frac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}}$;

hence we have the following rule:

From half the sum of the three sides subtract each of the two sides which contain the required angle.

Add the log sines of these two remainders, and the complement logs of the sines of the sides which contain the angle.

Half the sum of these four logarithms will give the log sine of half the required angle. Thus:

79° 17′ 14″
110
58

2)247 17 14

123 38 37 = ½ sum of the three sides.
79 17 14

44 21 23 first remainder — log sin = 9,8445513

123 38 37
58

65 38 37 second remainder — log sin = 9,9595173
comp log sin 58° — 0,0715795
comp log sin 79° 17′ 14″ — 0,0076359

2)19,8832840

log sin = 60° 57′ 28″ = 9,9416420
2

121 54 56 equals the required angle A.

By a similar operation the angles B and C may be found; but when one angle is known, the other two are easily determined by Art. 13, page 6.

CASE 2. Given the angle $A = 32° 20' 30''$, the side $b = 72° 10' 20''$, and the side $a = 78° 59' 10''$, to find B, C and c.

Here by page 6, $\sin B = \dfrac{\sin A \,.\, \sin b}{\sin a}$

$$\log \sin B = \log \sin A + \log \sin b - \log \sin a$$

$$\begin{aligned} \log \sin A &= 9{\cdot}7283269 \\ \log \sin b &= 9{\cdot}9786283 \\ \hline & 19{\cdot}7069552 \\ \log \sin a &= 9{\cdot}9919261 \\ \hline \log \sin B &= 9{\cdot}7150291 \end{aligned}$$

$$\therefore B = 31° 15' 15''.$$

By page 9, equation (16)

$$\cot \tfrac{1}{2} C = \tan \tfrac{1}{2} (A + B) \frac{\cos \tfrac{1}{2} (a + b)}{\cos \tfrac{1}{2} (a - b)}$$

$$\log \cot \tfrac{1}{2} C =$$

$$\log \tan \tfrac{1}{2} (A + B) + \log \cos \tfrac{1}{2} (a + b) - \log \cos \tfrac{1}{2} (a - b)$$

$$\begin{aligned} \log \tan \tfrac{1}{2} (A + B) &= \log \tan 31° 47' 52'' = 9{\cdot}7217470 \\ \log \cos \tfrac{1}{2} (a + b) &= \log \cos 75° 34' 45'' = 9{\cdot}3962727 \\ \hline & 19{\cdot}1180197 \\ \log \cos \tfrac{1}{2} (a - b) &= \log \cos 3° 24' 25'' = 9\ 9992318 \\ \hline \log \cot \tfrac{1}{2} C &= 9{\cdot}1187879 \end{aligned}$$

$$\therefore \tfrac{1}{2} C = 82° 30' 39'';$$

$$\text{or } C = 165° 1' 18''.$$

We might find c from the equation

$$\sin c = \sin a \,.\, \frac{\sin C}{\sin A},$$

but we can find it directly from Napier's Analogies.

By page 9, equation (14), we have

$$\tan \tfrac{1}{2} c = \tan \tfrac{1}{2} (a + b) \cdot \frac{\cos \tfrac{1}{2} (A + B)}{\cos \tfrac{1}{2} (A - B)};$$

$\therefore$ log tan $\frac{1}{2} c =$
log tan $\frac{1}{2} (a + b)$ + log cos $\frac{1}{2} (A + B)$ − log cos $\frac{1}{2} (A - B)$

log tan $\frac{1}{2} (a + b)$ = log tan 75° 34′ 45″ = 10·5898236
log cos $\frac{1}{2} (A + B)$ = log cos 31° 47′ 52″ = 9·9293745

20·5191981
log cos $\frac{1}{2} (A - B)$ = log cos 1° 5′ 15″ = 9·9999218

log tan $\frac{1}{2} c$ = 10·5192763

$\frac{1}{2} c = 73° 10′ 10″$

$\therefore c = 146° 20′ 20″.$

Case 3. Given C = 30° 45′ 28″; $a = 84° 14′ 29″$; $b = 44° 13′ 45″$, the two sides and the included angle, to find A, B, c.

By Napier's Analogies, page 9, equations (16) and (17),

$$\tan \tfrac{1}{2} (A + B) = \cot \tfrac{1}{2} C \cdot \frac{\cos \tfrac{1}{2} (a - b)}{\cos \tfrac{1}{2} (a + b)}$$

$$\text{and } \tan \tfrac{1}{2} (A - B) = \cot \tfrac{1}{2} C \cdot \frac{\sin \tfrac{1}{2} (a - b)}{\sin \tfrac{1}{2} (a + b)}$$

$\frac{1}{2}$ C = 18° 22′ 44″ log cot = 10·4785395
$\frac{1}{2} (a - b)$ = 20° 0° 22″ log cos = 9·9729690

20·4515085
$\frac{1}{2} (a + b)$ = 64° 14′ 7″ log cos = 9·6381663

$\therefore$ log tan $\frac{1}{2} (A + B)$ = 10·8133422

$\therefore \frac{1}{2} (A + B) = 81° 15′ 44″.41.$

A — B determined.

$\frac{1}{2}$ C = 18° 22′ 44″ log cot = 10·4785395
$\frac{1}{2}(a-b)$ = 20 0 22 log sin = 9·5341789

20·0127184

$\frac{1}{2}(a+b)$ = 64 14 7 log sin = 9·9545255

10·0581929

$\frac{1}{2}$ (A — B) = 48° 49′ 38″.

A and B determined.

$\frac{1}{2}$(A+B)=81° 15′ 44″·41
$\frac{1}{2}$(A—B)=48 49 38

∴ A =130° 5′ 22″·41
B = 32 26 6·41

c determined.

log sin 36° 45′ 28″ = 9·7770158
log sin 44 13 45 = 9·8435629

19·6205787
log sin 32 26 6 = 9·7294422

∴ log sin c = 9·8911365

∴ c = 51° 6′ 12″.

c may be found directly, without finding A and B, by the following method :—

Since $\cos C = \frac{\cos c - \cos a \,.\, \cos b}{\sin a \,.\, \sin b}$, page 5,

$\therefore \cos c = \cos a \,.\, \cos b + \sin a \,.\, \sin b \,.\, \cos C$;

but $\cos C = 1 - \text{ver. sin } C$,

$\therefore \cos c = \cos a \,.\, \cos b + \sin a \,.\, \sin b - \sin a \,.\, \sin b \,.\, \text{ver. sin } C$,

$= \cos(a-b) - \sin a \,.\, \sin b \,.\, \text{ver. sin } C$;

$\therefore 1 - \cos c$, or $2\sin^2\frac{c}{2} = \text{ver. sin}(a-b) + \sin a \,.\, \sin b \,.\, \text{ver. sin } C$,

$$= \text{ver. sin}(a-b)\left(1 + \frac{\sin a \,.\, \sin b \,.\, \text{ver. sin } C}{\text{ver. sin}(a-b)}\right).$$

Let $\tan^2\theta = \frac{\sin a \,.\, \sin b \,.\, \text{ver. sin } C}{\text{ver. sin}(a-b)}$;

which in logarithms is $2 \log \tan \theta =$

$\log \sin a + \log \sin b + \log \text{ver. sin } C - \log \text{ver. sin } (a-b) \ldots [a]$

then $2 \sin^2 \frac{c}{2} = \text{ver. sin } (a-b) \cdot \sec^2 \theta$, and

$\log 2 + 2 \log \sin \frac{c}{2} = \log \text{ver. sin } (a-b) + 2 \log \sec \theta - 10 \ldots [b]$

c computed independently of A and B.

Finding the auxiliary angle θ by the form $[a]$.

$a = 84° 14' 29''$	sin =	9·9978028
$b = 36\ 13\ 45$	sin =	9·8435629
$C = 36\ 45\ 28$	ver. sin =	9·2984762
		29·1398419
$a - b = 40\ 0\ 44$	ver. sin =	9·3693878
$\therefore$ 2 log tan θ	=	19·7704541
and log tan θ	=	9·8852270

Case 4. Given $c = 50° 6' 20''$; $A = 129° 58' 30''$; $B = 34° 29' 30''$; to find a, b, C.

By equations (14) and (15), page 9,

$$\tan \tfrac{1}{2}(a+b) = \tan \tfrac{1}{2} c \cdot \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}$$

$$\tan \tfrac{1}{2}(a-b) = \tan \tfrac{1}{2} c \cdot \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}$$

$\left.\begin{aligned} \tfrac{1}{2}(A+B) &= 82° 14' \\ \tfrac{1}{2}(A-B) &= 47° 44' 30'' \\ \tfrac{1}{2} c &= 25° 3' 10'' \end{aligned}\right\}$

$\log \tan \frac{1}{2}(a+b) =$

$\log \tan \frac{1}{2} c + \log \cos \frac{1}{2}(A-B) - \log \cos \frac{1}{2}(A+B)$

$\log \tan \frac{1}{2} c = \log \tan 25° 3' 10''$	=	9·6697162
$\log \cos \frac{1}{2}(A-B) = \log \cos 47° 44' 30''$	=	9·8276758
		19·4973920
$\log \cos \frac{1}{2}(A+B) = \log \cos 82° 14'$	=	9·1307812
$\log \tan \frac{1}{2}(a+b)$	=	10·3666108

$$\therefore \tfrac{1}{2}(a+b) = 66^\circ\ 44'\ 10''$$

$$\log\tan\tfrac{1}{2}(a-b) =$$
$$\log\tan\tfrac{1}{2}c + \log\sin\tfrac{1}{2}(\mathrm{A}-\mathrm{B}) - \log\sin\tfrac{1}{2}(\mathrm{A}+\mathrm{B})$$

$\log\tan\frac{1}{2}c = \log\tan 25^\circ\ 3'\ 10'' = 9{\cdot}6697162$
$\log\sin\frac{1}{2}(\mathrm{A}-\mathrm{B}) = \log\sin 47^\circ\ 44'\ 30'' = 9{\cdot}8693023$

$19{\cdot}5390185$
$\log\sin\frac{1}{2}(\mathrm{A}+\mathrm{B}) = \log\sin 82^\circ\ 14' = 9{\cdot}9959977$

$\log\tan\frac{1}{2}(a-b) = 9{\cdot}5430208$

$$\tfrac{1}{2}(a-b) = 19^\circ\ 14'\ 50''$$
$$\tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b) = a$$
$$\tfrac{1}{2}(a+b) - \tfrac{1}{2}(a-b) = b$$

$66^\circ\ 44'\ 10''$
$19^\circ\ 14'\ 50''$

$85^\circ\ 59'\ 0''$

$47^\circ\ 29'\ 20''$

$\therefore a = 85^\circ\ 59'$ and $b = 47^\circ\ 29'\ 20''$.

To find C.

$$\frac{\sin\mathrm{A}}{\sin a} = \frac{\sin\mathrm{C}}{\sin c}$$

$$\text{or, } \sin\mathrm{C} = \sin c \,.\, \frac{\sin\mathrm{A}}{\sin a}$$

$$\log\sin\mathrm{C} = \log\sin c + \log\sin\mathrm{A} - \log\sin a$$

$\log\sin\mathrm{A} = \log\sin 129^\circ\ 58'\ 30'' = 9{\cdot}8844129$
$\log\sin c = \log\sin 50^\circ\ 6'\ 20'' = 9{\cdot}8849241$

$19{\cdot}7693370$
$\log\sin a = \log\sin 85^\circ\ 59' = 9{\cdot}9989319$

$\log\sin\mathrm{C} = 9{\cdot}7704051$

$$\therefore \mathrm{C} = 36^\circ\ 6'\ 50''.$$

CASE 5. Given the angles $A = 70° 39'$; $B = 48° 36'$; $a = 89° 16' 53''$, to find the rest.

By page 21, we have $\sin b = \dfrac{\sin a \sin B}{\sin A}$

$$\log \sin b = \log \sin a + \log \sin B - \log \sin A$$

$$\begin{aligned}
\log \sin a = \log \sin 89° 16' 53'' &= 9{\cdot}9999658 \\
\log \sin B = \log \sin 48° 36' &= 9{\cdot}8751256 \\
&\overline{19{\cdot}8750914} \\
\log \sin A = \log \sin 70° 39' &= 9{\cdot}9747475 \\
&\overline{\phantom{19{\cdot}8750914}} \\
\log \sin b &= 9{\cdot}9003439
\end{aligned}$$

$$\therefore b = 52° 39' 4''$$

$\sin b = \sin (180 - b) = \sin 127° 20' 56''$; but since $A > B$, a must be greater than b, hence b cannot be $127° 20' 56''$.

To find c.

By Napier's Analogies

$$\tan \tfrac{1}{2} c = \tan \tfrac{1}{2} (a + b) \cdot \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)}$$

$$\log \tan \tfrac{1}{2} c =$$

$$\log \tan \tfrac{1}{2} (a + b) + \log \cos \tfrac{1}{2} (A + B) - \log \cos \tfrac{1}{2} (A - B)$$

$$\begin{aligned}
\log \tan \tfrac{1}{2} (a + b) = \log \tan 70° 57' 59'' &= 10{\cdot}4622011 \\
\log \cos \tfrac{1}{2} (A + B) = \log \cos 59° 37' 30'' &= 9{\cdot}7038563 \\
&\overline{20{\cdot}1660574} \\
\log \cos \tfrac{1}{2} (A - B) = \log \cos 11° 1' 30'' &= 9{\cdot}9919097 \\
&\overline{\phantom{20{\cdot}1660574}} \\
\log \tan \tfrac{1}{2} c &= 10{\cdot}1741477
\end{aligned}$$

$$\therefore \tfrac{1}{2} c = 56° 11' 29'',$$

$$\text{or } c = 112° 22' 58''.$$

By equation (16), page 9, we have

$$\cot \tfrac{1}{2} C = \tan \tfrac{1}{2} (A + B) \frac{\cos \frac{1}{2} (a + b)}{\cos \frac{1}{2} (a - b)}$$

$\log \cot \frac{1}{2} C \log =$

$\log \tan \frac{1}{2}(A + B) + \log \cos \frac{1}{2}(a + b) - \log \cos \frac{1}{2}(a - b)$

$\log \tan \frac{1}{2}(A + B) = \log \tan 59° 37' 30'' = 10{\cdot}2320208$

$\log \cos \frac{1}{2}(a + b) = \log \cos 70° 57' 59'' = 9{\cdot}5133811$

$19{\cdot}7454019$

$\log \cos \frac{1}{2}(a - b) = \log \cos 18° 18' 54'' = 9{\cdot}9774233$

$\log \cot \frac{1}{2} C = 9{\cdot}7679786$

$\therefore \frac{1}{2} C = 59° 37' 30'',$

or $C = 119° 15'.$

Case 6. Given the three angles,

angle A = 120° 54′ 56″
angle B = 50
angle C = 62 34 6

} to find the rest.

By page 22, $\cos \frac{1}{2} a = \sqrt{\frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}}$;

hence the following rule.

To find the side BC.

From half the sum of the three angles take each of the angles next the required side.

Add the log cosines of these two remainders, and the comp. log of the sines of each of the adjoining angles.

Half the sum of these four logarithms will give the cosine of half the required side; thus

121°	54′	56″		
50				
62	34	6		
2) 234	29	2		
117	14	31		
62	34	6		
54	40	25	first rem.	cos 9,7621032
67	14	31	second rem.	cos 9,5875321
			comp. log sin 50°	0,1157460
			comp. log sin 62° 34′ 6″	0,0518018
				2) 19,5171831
			cosine 55°	9,7585915
			2	

110° = the required side BC.

By the same method the other sides may be found; but one side being known (with the angles) the rest are most readily found by Art. 14, page 6.

CHAPTER III.

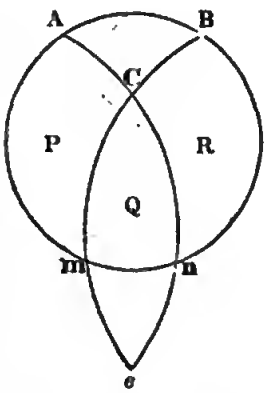
41. THE surface of the sphere included between the arcs D M, D N is proportional to the angle N D M or the arc M N. See fig. page 1.

If the circumference be divided into equal parts as M N, and great circles be drawn from D through the points M, N, the portions of the surface, such as N D M, are all similar and equal, hence if F M contains N M, n times, or if F M $= n$ times N M, the surface F D M will be n times N D M.

When D M coincides with D G, the angle F D G or its measure F M G $= 180$; hence if S $=$ whole surface of the sphere, and if A be the angle N D M which is measured by the arc N M, the surface $\text{N D M} = \frac{\text{S}}{4} \cdot \frac{\text{A}}{180}$, but S $=$ area of 4 great circles of the sphere. Hall's Diff. Cal., page 370.

$$\therefore \text{S} = 4\,\pi\,r^2 = 4\,\pi \text{ when radius is unity,}$$

$$\text{or } \frac{\text{S}}{4} = \pi = 180.$$



The measure of the surface of a spherical triangle is the difference between the sum of its three angles and two right angles.

Let the triangle be A B C, a, b, c, representing the magnitudes of the angles at A, B, C; let P=surface B Cm B, Q=m Cn m, R=A Cn A; produce the arcs Cm, Cn, till they meet at e (which will be on the hemisphere opposite to that represented by A B$m$$n$A), then each of the angles at C and e equals the angle of the planes in which the arcs C$m$$e$, C$n$$e$, lie; therefore the angles at C and e are equal.

Again, the semicircles A C m, C m e; B C n, C n e are equal; or, A C + C m = C m + m e, and ∴ A C = m e, and B C = n e; and the triangle m e n = the triangle A B C; let x = its area, then, by last article,

$$x + \mathrm{P} = \frac{\mathrm{S}}{2} \cdot \frac{a}{180}$$

$$x + \mathrm{Q} = \frac{\mathrm{S}}{2} \cdot \frac{c}{180}, \text{ and } x + \mathrm{P} + \mathrm{Q} + \mathrm{R} = \frac{\mathrm{S}}{2}$$

$$x + \mathrm{R} = \frac{\mathrm{S}}{2} \cdot \frac{b}{180};$$

consequently, by addition,

$$2x + (x + \mathrm{P} + \mathrm{Q} + \mathrm{R}) \text{ or } 2x + \frac{\mathrm{S}}{2} = \frac{\mathrm{S}}{2} \cdot \left(\frac{a+b+c}{180}\right);$$

$$\therefore x = \frac{\mathrm{S}}{4\,(180)}(a+b+c) - \frac{\mathrm{S}}{4} = a + b + c - 180°;$$

$$\text{or } r^2\,(a + b + c - 180°).$$

Hence the area of a spherical triangle is equal to the excess of the sum of its three angles above two right angles, which is called the *spherical excess.*

The late Professor Woodhouse, in his able work on Trigonometry, observes that—"This expression for the value of the area was merely a speculative truth, and continued barren for more than 150 years, till 1787, when General Roy employed it in correcting the spherical angles of observation made in the great Trigonometrical Survey."

In a biographical sketch of the life of Isaac Dalby, late Professor of Mathematics at the Royal Military College, Sandhurst, in Leybourne's Mathematical Repository, it is stated that he had sent some years previously to his death an account of the principal events of his life after reaching maturity. The following is a quotation from himself given in the above-named excellent periodical:—

"General Roy's account of this measurement is in the Philosophical Transactions; but it is not altogether what it ought to have been. His description of the apparatus, detail of occurrences, &c., are all well enough; but he should not have meddled with the mathematical part, *for his knowledge did not extend beyond Plane Trigonometry.* I drew up

the computations in that form which I thought the most proper for publication, but he was continually making alterations. *He did not even understand the rule I made use of for finding the excess of the sum of the three angles of a spherical triangle above* 180° (*which since that time has been quoted as General Roy's theorem*), *and would not insert it until he had consulted the Hon. Henry Cavendish.* For conducting the business in the field, however, few persons could have been better qualified than the General.

"I believe he was the best topographer in England, and knew the situation of every barrow, cairn, and hillock in Great Britain. He had something of an observatory in the upper part of his dwelling, and could regulate a clock or watch by means of transits. In fact, he was ready enough at calculations which depended merely on the use of the tables. *But the rules which he published for measuring the heights of the barometer all came from Mr. Ramsden.*"

A note is given to this extract in the Repository, which is as follows:—"It is not until very recently that Mr. Dalby has had justice done him with regard to this ingenious rule. At page 138 of the new edition of Vol. III. of Hutton's Course of Mathematics, published by Dr. Gregory in 1827, we find this note:—'This is commonly called General Roy's rule, and given by him in the Philosophical Transactions for 1790, p. 171; it is, however, due to the late Mr. Isaac Dalby, who was then General Roy's assistant in the Trigonometrical Survey, and for several years the entire conductor of the mathematical department.'"

FURTHER DEVELOPMENTS CONCERNING THE SPHERICAL EXCESS.

42. Let the radius of the sphere be unity, π the semicircumference of a great circle; a, b, c, the three sides of a spherical triangle; A, B, C, the arcs of a great circle that measure the opposite angles.

Let the spherical excess $A + B + C - \pi = S$.

The area of the spherical triangle is equal to the arc S multiplied by the radius, and is therefore represented by S.

Now, by Napier's Analogies,

$$\tan \tfrac{1}{2}(A + B) = \cot \tfrac{1}{2} C \cdot \frac{\cos \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2}(a + b)}$$

$$\tan \{\tfrac{1}{2}(A + B) + \tfrac{1}{2} C\} = \frac{\tan \tfrac{1}{2}(A + B) + \tan \tfrac{1}{2} C}{1 - \tan \tfrac{1}{2}(A + B) \tan \tfrac{1}{2} C}$$

$$= \frac{\cot \tfrac{1}{2} C \cdot \dfrac{\cos \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2}(a + b)} + \tan \tfrac{1}{2} C}{1 - \cot \tfrac{1}{2} C \cdot \dfrac{\cos \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2}(a + b)} \cdot \tan \tfrac{1}{2} C}$$

$$= \frac{\cot \tfrac{1}{2} C \cdot \dfrac{\cos \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2}(a + b)} + \tan \tfrac{1}{2} C}{1 - \dfrac{\cos \tfrac{1}{2}(a - b)}{\cos \tfrac{1}{2}(a + b)}}$$

$$= \frac{\cot \tfrac{1}{2} C \cdot \cos \tfrac{1}{2}(a - b) + \tan \tfrac{1}{2} C \cdot \cos \tfrac{1}{2}(a + b)}{\cos \tfrac{1}{2}(a + b) - \cos \tfrac{1}{2}(a - b)}$$

$$\frac{\dfrac{1 + \cos C}{\sin C} \cos \tfrac{1}{2}(a - b) + \dfrac{1 - \cos C}{\sin C} \cos \tfrac{1}{2}(a + b)}{\cos \tfrac{1}{2}(a + b) - \cos \tfrac{1}{2}(a - b)} =$$

(by expanding the cosines and reducing)

$$\frac{1}{\sin C} \cdot \frac{\cos \tfrac{1}{2} a \cos \tfrac{1}{2} b + \sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \cos C}{-\sin \tfrac{1}{2} a \sin \tfrac{1}{2} b}$$

$$= \frac{-\cot \tfrac{1}{2} a \cot \tfrac{1}{2} b - \cos C}{\sin C}$$

but $\tan \tfrac{1}{2}(A + B + C) = \tan \tfrac{1}{2}(S + 180) = -\cot \tfrac{1}{2} S$;

$$\therefore \cot \tfrac{1}{2} S = \frac{\cot \tfrac{1}{2} a \cot \tfrac{1}{2} b + \cos C}{\sin C}.$$

This equation, which is very simple, enables us to find the area of a spherical triangle when the two sides and the included angle are given.

To find the area of a spherical triangle in terms of the three sides,

$$\cot \tfrac{1}{2} S = \frac{\cot \frac{1}{2} a \cot \frac{1}{2} b + \cos C}{\sin C}$$

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}$$

$$\text{and } \cot \tfrac{1}{2} a = \frac{1+\cos a}{\sin a}, \ \cot \tfrac{1}{2} b = \frac{1+\cos b}{\sin b};$$

$$\therefore \cos C + \cot \tfrac{1}{2} a \cot \tfrac{1}{2} b = \frac{1 + \cos a + \cos b + \cos c}{\sin a \sin b}$$

$$1 + \cos C = \frac{\cos c - \cos (a+b)}{\sin a \sin b} = \frac{2 \sin \frac{+b+c}{2} \sin \frac{a+b-c}{2}}{\sin a \sin b}$$

$$1 - \cos C = \frac{\cos (a-b) - \cos c}{\sin a \sin b} = \frac{\frac{(2 \sin a+b+c)}{2} \sin \frac{b+c-a}{2}}{\sin a \sin b}$$

Multiplying these two equations and extracting the root,

$$\sin C = 2 \frac{\sqrt{\sin \frac{a+b+c}{2} \sin \frac{a+b-c}{2} \sin \frac{a+c-b}{2} \sin \frac{b+c-a}{2}}}{\sin a \sin b}$$

By substituting these values we have

$$\cot \tfrac{1}{2} S = 2 \frac{1+\cos a+\cos b+\cos c}{\sqrt{\sin \frac{a+b+c}{2} \sin \frac{a+b-c}{2} \sin \frac{a+c-b}{2} \sin \frac{b+a-b}{2}}}$$

This solves the problem, but it can be put into a simpler form, and one that is adapted to logarithmic computation.

Resuming the formula

$$\cot \tfrac{1}{2} S = \frac{\cot \tfrac{1}{2} a \cot \tfrac{1}{2} b + \cos C}{\sin C},$$

$$1 + \cot^2 \tfrac{1}{2} S = \frac{1}{\sin^2 \tfrac{1}{2} S} =$$

$$\frac{\cot^2 \tfrac{1}{2} a \cot^2 \tfrac{1}{2} b + 2 \cot \tfrac{1}{2} a \cot \tfrac{1}{2} b \cos C + 1}{\sin^2 C},$$

by multiplying both sides of the value of cos C by $2 \cot \frac{1}{2} a \cot \frac{1}{2} b$, we have

$$2 \cot \tfrac{1}{2} a \cot \tfrac{1}{2} b \cos C = \frac{\cos c - \cos a \cos b}{2 \sin^2 \tfrac{1}{2} a \sin^2 \tfrac{1}{2} b},$$

putting in the numerator for $\cos c$, $\cos a$, $\cos b$, their values $1 - 2 \sin^2 \frac{1}{2} c$, $1 - 2 \sin^2 \frac{1}{2} a$, $1 - 2 \sin^2 \frac{1}{2} b$, we shall have

$$2 \cot \tfrac{1}{2} a \cot \tfrac{1}{2} b \cos C = \frac{\sin^2 \tfrac{1}{2} a + \sin^2 \tfrac{1}{2} b - \sin^2 \tfrac{1}{2} c}{\sin^2 \tfrac{1}{2} a \sin^2 \tfrac{1}{2} b} - 2.$$

Also, $\cot^2 \frac{1}{2} a \cot^2 \frac{1}{2} b = \frac{1 - \sin^2 \frac{1}{2} a}{\sin^2 \frac{1}{2} a} \cdot \frac{1 - \sin^2 \frac{1}{2} b}{\sin^2 \frac{1}{2} b} =$

$$\frac{1 - \sin^2 \tfrac{1}{2} a - \sin^2 \tfrac{1}{2} b}{\sin^2 \tfrac{1}{2} a \sin^2 \tfrac{1}{2} b} + 1.$$

Substituting all these values, we have

$$\frac{1}{\sin^2 \tfrac{1}{2} S} = \frac{1 - \sin^2 \tfrac{1}{2} c}{\sin^2 \tfrac{1}{2} a \sin^2 \tfrac{1}{2} b \sin^2 C}, \text{ or}$$

$$\sin \tfrac{1}{2} S = \frac{\sin \tfrac{1}{2} a \sin \tfrac{1}{2} b \sin C}{\cos \tfrac{1}{2} c}.$$

Substituting for sin C its value, we have

$\sin \frac{1}{2} S =$

$$\frac{\sqrt{\left(\sin \frac{a+b+c}{2} \sin \frac{a+b-c}{2} \sin \frac{a+c-b}{2} \sin \frac{b+c-a}{2}\right)}}{2 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}$$

which expression is adapted to logarithmic computation.

If we multiply this equation by $\cot \frac{1}{2} S$ we have

$$\cos \tfrac{1}{2} S = \frac{1 + \cos a + \cos b + \cos c}{4 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c} =$$

$$\frac{\cos^2 \frac{1}{2} a + \cos^2 \frac{1}{2} b + \cos^2 \frac{1}{2} c - 1}{2 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c};$$

From this we have,

$$\frac{1 - \cos \frac{1}{2} S}{\sin \frac{1}{2} S} \text{ or } \tan \tfrac{1}{4} S =$$

$$\frac{1 - \cos^2 \frac{1}{2} a - \cos^2 \frac{1}{2} b - \cos^2 \frac{1}{2} c + 2 \cos \frac{1}{2} a \cos \frac{1}{2} b \cos \frac{1}{2} c}{\sqrt{\left\{\sin \frac{a+b+c}{2} . \sin \frac{a+b-c}{2} . \sin \frac{a+c-b}{2} . \sin \frac{b+c-a}{2}\right\}}}$$

The numerator of this expression can be put under the form $(1 - \cos^2 \frac{1}{2} a)(1 - \cos^2 \frac{1}{2} b) - (\cos \frac{1}{2} a \cos \frac{1}{2} b - \cos \frac{1}{2} c)^2$ which may be decomposed into the factors

$$\sin \tfrac{1}{2} a \sin \tfrac{1}{2} b + \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b - \cos \tfrac{1}{2} c \text{ and}$$

$$\sin \tfrac{1}{2} a \sin \tfrac{1}{2} b - \cos \tfrac{1}{2} a \cos \tfrac{1}{2} b + \cos \tfrac{1}{2} c.$$

These reduce ultimately, the first to $\cos (\frac{1}{2} a - \frac{1}{2} b) - \cos \frac{1}{2} c =$

$$2 \sin \frac{a+c-b}{4} \sin \frac{b+c-a}{4},$$

the second to $\cos \frac{1}{2} c - \cos (\frac{1}{2} a + \frac{1}{2} b) =$

$$2 \sin \frac{a+b+c}{4} \sin \frac{a+b-c}{4};$$

$$\therefore \tan \tfrac{1}{4} S =$$

$$\frac{4 \sin \frac{a+b+c}{4} \sin \frac{a+b-c}{4} \sin \frac{a+c-b}{4} \sin \frac{b+c-a}{4}}{\sqrt{\left\{\sin \frac{a+b+c}{2} \sin \frac{a+b-c}{2} \sin \frac{a+c-b}{2} \sin \frac{b+c-a}{2}\right\}}}$$

$$\therefore \tan \tfrac{1}{4} S =$$

$$\sqrt{\tan \frac{a+b+c}{4} \tan \frac{a+b-c}{4} \tan \frac{a+c-b}{4} \cdot \tan \frac{b+c-a}{4}}.$$

This elegant formula is due to Simon Lhuillier. See Legendre's Geometry, page 319.

GIRARD'S THEOREM.

43. By page 45,

$a + b + c - 180° = \frac{180°}{\pi r^2} x$, or reducing a, b, c to seconds,

the excess in seconds $= \frac{180° \times 60 \times 60}{\pi r^2} x.$

Now, on the earth's surface, the length of 1° taking a mean measurement $= (60859{\cdot}1) \times 6$ feet, and an arc $=$ radius $= \frac{360}{2\pi}$;

$\therefore (60859{\cdot}1) \times 6 \times \frac{360}{2\pi} =$ radius of the earth in feet;

$\therefore$ the excess in seconds $= x \cdot \frac{2\pi}{360} \cdot \frac{60 \times 60}{6^2 \times (60859{\cdot}1)^2} =$

$$\frac{2\pi \times 10}{36 \times (60859{\cdot}1)^2} \cdot x.$$

$$\log \text{.excess} =$$

$$\log x - \{2 (\log 6 + \log 60859{\cdot}1) - \log 2\pi \times 10\}$$

$$= \log x - (11{\cdot}1249536 - 1{\cdot}7981799)$$

$$= \log x - 9{\cdot}3267737.$$

Hence the following rule:—

From the logarithm of the area of the triangle, considered as a plane one, in feet, subtract the constant logarithm 9·3267737, and the remainder is the logarithm of the excess above 180°, in seconds nearly.

Ex. Hanger Hill Tower ..... (*a*) Observed angles. 42° 2′ 32″

Hampton Poor-house ..... (*b*) 67 55 39

King's Arbour..... (*c*) 70 1 48

Distance from (*a*) to (*b*) = 38461·12 feet,

from (*a*) to (*c*) = 24704·7.

Taking the distance from (*a*) to (*c*) for the base of the triangle, the perpendicular on the base will be 38461·12 × sin 42° 2′ 32″, and therefore the area of the triangle

$$= \frac{\text{base} \times \text{perpendicular}}{2}$$

$$= 24704{\cdot}7 \times 19230{\cdot}56 \times \sin 42^\circ\, 2'\, 32'',$$

$$\log \text{area} = \log 24704{\cdot}7 + \log 19230{\cdot}56 + \log \sin 42^\circ\, 2'\, 32'' - 10$$

$$= 4{\cdot}3927761 + 4{\cdot}2839906 + 9{\cdot}8258661$$

$$= 8{\cdot}5026328 = \text{logarithm of the area in feet;}$$

$$\text{hence, } 8{\cdot}5026328 - 9{\cdot}3267737 = -1{\cdot}1758591;$$

the corresponding natural number is ·14992, the spherical excess in seconds.

LEGENDRE'S SOLUTION OF SPHERICAL TRIANGLES WHOSE SIDES ARE VERY SMALL COMPARED WITH THE RADIUS OF THE SPHERE.

44. When the sides a, b, c, are very small with respect to the radius of the sphere, the proposed triangle is very little different from a rectilinear triangle, and, considering it as such, we can have a first solution approximately, but we neglect in this manner the excess of the sum of the three angles above 180°. To have a solution more approximate, we must take

into account this excess, and this we can do very easily by means of a general principle which we proceed to demonstrate.

Let r be the radius of the sphere upon which the triangle is situated, and if we imagine a similar triangle upon the sphere whose radius is unity, the sides of this triangle will be $\frac{a}{r}, \frac{b}{r}, \frac{c}{r}$; and we shall have cos A =

$$\frac{\cos\frac{a}{r} - \cos\frac{b}{r}\cos\frac{c}{r}}{\sin\frac{b}{r}\sin\frac{c}{r}};$$

but since r is very great with respect to a, b, c, we shall have approximately,

$$\cos\frac{a}{r} = 1 - \frac{a^2}{2r^2} + \frac{a^4}{2.3.4r^4};$$

$$\cos\frac{b}{r} = 1 - \frac{b^2}{2r^2} + \frac{b^4}{2.3.4r^4},$$

$$\cos\frac{c}{r} = 1 - \frac{c^2}{2r^2} + \frac{c^4}{2.3.4r^4},$$

$$\sin\frac{b}{r} = \frac{b}{r} - \frac{b^3}{2.3r^3};$$

$$\sin\frac{c}{r} = \frac{c}{r} - \frac{c^3}{2.3r^3}$$

Substituting these values in the above equation,

$$\cos A = \frac{\frac{b^2+c^2-a^2}{2r^2} + \frac{a^4-b^4-c^4}{24r^4} - \frac{b^2c^2}{4r^4}}{\frac{bc}{r^2}\left(1 - \frac{b^2}{6r^2} - \frac{c^2}{6r^2}\right)}$$

Multiply numerator and denominator by $1 + \frac{b^2+c^2}{6r^2}$ and reducing

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,b\,c} + \frac{a^4 + b^4 + c^4 - 2\,a^2\,b^2 - 2\,a^2\,c^2 - 2\,b^2\,c^2}{24\,b\,c\,r^2}$$

Let A′ be the angle opposite to the side a in the rectilinear triangle of which the sides are equal in length to the arcs a, b, c, we shall have

$$\cos A' = \frac{b^2+c^2-a^2}{2\,b\,c}, \text{ and}$$

$$4\,b^2\,c^2 \sin^2 A' = 2\,a^2\,b^2 + 2\,a^2\,c^2 + 2\,b^2\,c^2 - a^4 - b^4 - c^4;$$

therefore $\cos A = \cos A' - \frac{b\,c}{6\,r^2} \cdot \sin {}^2A'$

let $A = A' + x$, we shall have rejecting the square of x,

$$\cos A = \cos A' - x \sin A',$$

$$\text{from whence we have } x = \frac{b\,c}{6\,r^2} \sin A',$$

and since x is of the second order with respect to $\frac{b}{r}$ and $\frac{c}{r}$, it follows that the result is exact to quantities of the fourth order, we shall then have

$$A = A' + \frac{b\,c}{6\,r^2} \sin A';$$

but $\frac{1}{2}\,b\,c \sin A'$ = the area of the rectilinear triangle, of which the three sides are a, b, c, do not differ sensibly from those of the proposed spherical triangle. Then, if either area be called α, we shall have

$$A = A' + \frac{\alpha}{3\,r^2}, \text{ or } A' = A - \frac{\alpha}{3\,r^2}.$$

$$\text{Similarly, } B' = B - \frac{\alpha}{3\,r^2};\ C' = C - \frac{\alpha}{3\,r^2},$$

hence there results,

$$A' + B' + C' \text{ or } 180 = A + B + C - \frac{\alpha}{r^2},$$

we can then consider $\frac{\alpha}{r^2}$ as the excess of the three angles of the spherical triangle above two right angles.

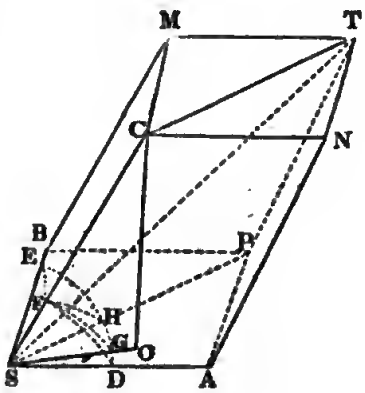
Hence we have the following rule:—

A spherical triangle being proposed, of which the sides are very small with respect to the radius of the earth, if from each of its angles one-third of the excess of the sum of its three angles above two right angles be subtracted, the angles so diminished may be taken for the angles of a rectilinear triangle, the sides of which are equal in length to those of the proposed spherical triangle, or in other terms:—The spherical triangle, whose sides are nearly rectilinear, of which the angles are A, B, C, and the opposite sides a, b, c, answer always to a rectilinear triangle whose sides are of the same length, a, b, c, and of which the opposite angles are $A - \frac{1}{3}\epsilon$; $B - \frac{1}{3}\epsilon$; $C - \frac{1}{3}\epsilon$; ϵ being the excess of the sum of the angles of the spherical triangle proposed above two right angles.

The excess ϵ, or $\frac{\alpha}{r^2}$, which is proportional to the area of the triangle, can always be calculated *à priori*, by the given parts of the spherical triangle considered as rectilinear. If the two sides, b, c, and the included angle A, are given, we shall have the area $\alpha = \frac{1}{2} b c \sin A$; if we have given the side a, and the two adjacent angles B, C, we shall have the area

$$\alpha = \tfrac{1}{2} a^2 \frac{\sin B \sin C}{\sin (B + C)}.$$

45. Given the three edges of a parallelopiped, and the angles between them, to find the solidity.



Let the edges $SA = f$, $SB = g$, $SC = h$, and the contained angles $ASB = \alpha$, $ASC = \mathcal{C}$, $BSC = \gamma$, if from the point C we let fall C O perpendicularly on the plane ASB then in right-angled triangle C S O; $CO = CS \sin CSO = h \sin CSO$, besides the surface of the

parallelogram ASBP $= f\ g \sin \alpha$. Therefore, if we call S the solidity of the parallelopiped ST, we shall have S $= f.\ g.\ h.\ \sin \alpha \sin$ CSO. We now proceed to find sin CSO. From the point S as a centre and radius unity, describe a spherical surface meeting the right lines SA, SB, SC, SO, in the points D, E, F, G, we shall have a triangle DEF, in which the arc FG is perpendicular to ED, since the plane CSO is perpendicular to ASB. Now, in the triangle DEF, where the three sides, DE $= \alpha$, DF $= \mathcal{C}$, EF $= \gamma$, we have

$$\cos \mathrm{E} = \frac{\cos \mathcal{C} - \cos \alpha \cos \gamma}{\sin \alpha \sin \gamma}, \text{ and}$$

$$\sin \mathrm{E} = \frac{\sqrt{1 - \cos^2 \alpha - \cos^2 \mathcal{C} - \cos^2 \gamma + 2 \cos \alpha \cos \mathcal{C} \cos \gamma}}{\sin \alpha \sin \gamma}$$

Then in the right-angled triangle EFG, sin GF or sin CSO $=$ sin E sin EF $=$ sin γ sin E.

$$\therefore \mathrm{S} = f.\ g.\ h.\ \sin \alpha \sin \gamma \sin \mathrm{E} =$$

$$f.\ g.\ h. \sqrt{1 - \cos^2 \alpha - \cos^2 \mathcal{C} - \cos^2 \gamma + 2 \cos \alpha \cos \mathcal{C} \cos \gamma}.$$

The expression under the radical is composed of the two factors, $\sin \alpha \sin \gamma + \cos \mathcal{C} - \cos \alpha \cos \gamma$, and $\sin \alpha \sin \gamma - \cos \mathcal{C} + \cos \alpha \cos \gamma$, the first $= \cos \mathcal{C} - \cos (\alpha + \gamma) =$

$$\frac{2 \sin \frac{\alpha + \mathcal{C} + \gamma}{2}}{} . \sin \frac{\alpha + \gamma - \mathcal{C}}{2};$$

the second $= \cos (\alpha - \gamma) - \cos \mathcal{C} =$

$$2 \sin \frac{\alpha + \mathcal{C} - \gamma}{2} . \sin \frac{\mathcal{C} + \gamma - \alpha}{2};$$

therefore, S $=$

$$2fgh \sqrt{\sin \frac{\alpha+\mathcal{C}+\gamma}{2} \sin \frac{\alpha+\mathcal{C}-\gamma}{2} \sin \frac{\alpha+\gamma-\mathcal{C}}{2} . \sin \frac{\mathcal{C}+\gamma-\alpha}{2}}$$

46. The same things being given as in the above to find the diagonal.

Let the diagonal of the base $SP = z$, and the required diagonal $ST = u$, the triangle ASP, in which $\cos SAP = -\cos \alpha$, we have $z^2 = f^2 + g^2 + 2fg \cos \alpha$, in like manner in the triangle TSP, in which $\cos TPS = -\cos CSP$; $u^2 = z^2 + h^2 + 2hz \cos CSP$.

We must now find $\cos CSP$, or of the arc FH.

Now, in the spherical triangle EFH, we have $\cos FH = \cos EF \cos EH + \sin EF \sin EH \cos E$, substituting the values $FF = \gamma$ and $\cos E = \dfrac{\cos \mathcal{C} - \cos \alpha \cos \gamma}{\sin \alpha \sin \gamma}$ it becomes

$$\cos FH = \cos \gamma \cos EH + \frac{\sin EH}{\sin \alpha}(\cos \mathcal{C} - \cos \alpha \cos \gamma) =$$

$$\frac{\sin EH \cos \mathcal{C}}{\sin \alpha} + \frac{\sin (\alpha - EH) \cos \gamma}{\sin \alpha} =$$

$$\frac{\sin EH \cos \mathcal{C} + \sin DH \cos \gamma}{\sin \alpha}.$$

Therefore $2hz \cos FH$, or $2hz \cos CSP =$

$$2h \cos \mathcal{C} \cdot \frac{z \sin EH}{\sin \alpha} + 2h \cos \gamma \cdot \frac{z \sin DH}{\sin \alpha};$$

but in the triangle BSP we have

$$BP = \frac{SP \sin BSP}{\sin SBP} \text{ and } BS = \frac{SP \sin BPS}{\sin SBP};$$

$$\text{which gives } \frac{z \sin EH}{\sin \alpha} = f \text{ and } \frac{z \sin DH}{\sin \alpha} = g;$$

$$\therefore 2hz \cos CSP = 2fh \cos \mathcal{C} + 2gh \cos \gamma.$$

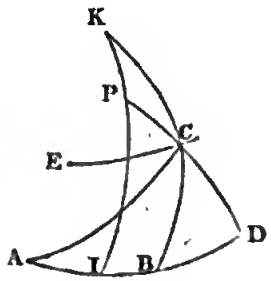
Hence the square of the required diagonal

$$u^2 = f^2 + g^2 + h^2 + 2fg \cos \alpha + 2fh \cos \mathcal{C} + 2gh \cos \gamma.$$

47. To determine a line on the surface of a sphere on which the vertices of all triangles of the same base and surface are situated.

Let A B C, be a spherical triangle, (one of those on the common base A B, $= c$); and the given surface $A + B + C - \pi = S$. Let I P K be an indefinite perpendicular on the middle of A B, having taken I P = a quadrant, P will be the pole of the arc A B, and the arc C D, drawn through the points P C, will be perpendicular to A B. Let I D $= p$, C D $= q$, the right-angled triangles, A C D, B C D, in which A C $= b$, B C $= a$, A D $= p + \frac{1}{2}c$, B D $= p - \frac{1}{2}c$, will give $\cos a = \cos q$, $\cos(p - \frac{1}{2}c)$, $\cos b = \cos q \cos(p + \frac{1}{2}c)$. But it was found, page 48, that

$$\cot \tfrac{1}{2} S = \frac{1 + \cos a + \cos b + \cos c}{\sin a \sin b \sin C}.$$



Substituting in this formula the values $\cos a + \cos b = 2 \cos q \cos p \cos \frac{1}{2}c$, $1 - \cos c = 2 \cos^2 \frac{1}{2} c$, $\sin b \sin C = \sin c \sin B = 2 \sin \frac{1}{2} c \cos \frac{1}{2} c \sin B$; there results,

$$\cot \tfrac{1}{2} S = \frac{\cos \frac{1}{2} c + \cos p \cos q}{\sin a \sin \frac{1}{2} c \sin B}.$$

Again, from the right-angled triangle B C D, $\sin a \sin B = \sin q$;

$$\therefore \cot \tfrac{1}{2} S = \frac{\cos \frac{1}{2} c + \cos p \,.\, \cos q}{\sin \frac{1}{2} c \sin q},$$

or, $\cos p \cos q = \cot \frac{1}{2} S \sin \frac{1}{2} c \sin q - \cos \frac{1}{2} c$. This is the relation between p and q which will determine the locus of all the points C.

Produce I P to K, let P K $= x$. Join K C, and let K C $= y$; in the triangle P K C where we have P C $= \frac{1}{2}\pi - q$, the angle K P C $= \pi - p$, the side K C will be found by the formula

$$\cos KC = \cos KPC \sin PK \sin PC + \cos PK \cos PC, \text{ or}$$

$$\cos y = \sin q \cos x - \sin x \cos p \cos q.$$

Substituting this instead of $\cos p \cos q$ the value

$$\cos \tfrac{1}{2} S \sin \tfrac{1}{2} c \sin q - \cos \tfrac{1}{2} c, \text{ there results}$$

$$\cos y = \sin x \cos \tfrac{1}{2} c + \sin q (\cos x - \sin x \cot \tfrac{1}{2} S \sin \tfrac{1}{2} c).$$

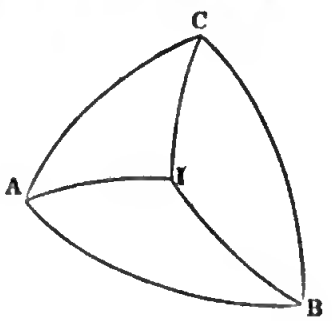
In which, if we take $\cos x - \sin x \cot \frac{1}{2} S \sin \frac{1}{2} c = 0$, or $\cot \frac{1}{2} S \sin \frac{1}{2} c = \cot x$, there will result $\cos y = \sin x \cos \frac{1}{2} c$, and thus a constant value of y is determined.

Therefore, if after having drawn the arc IP perpendicular to the middle of the base AB, and beyond the pole the part PK such that $\cot PK = \cot \frac{1}{2} S \sin \frac{1}{2} c$, all the vertices of the triangles on the same base c, and of the same surface S, will be situated on the small circle described from K as a pole at the distance KC, such that $\cos KC = \sin PK \cos \frac{1}{2} c$. This is Lexell's theorem.

48. Given the three sides, $BC = a$, $AC = b$, $AB = c$, to find the position of the point I, the pole of the circle circumscribing the triangle ABC.

Let the angle $ACI = x$, and the arc $AI = CI = BI = \varphi$; in the triangles CAI, CBI, we shall have by the equation—

$$\cos x = \frac{\cos \varphi - \cos b \cos \varphi}{\sin b \sin \varphi} =$$



$$\frac{1 - \cos b}{\sin b} \cot \varphi =$$

$$\frac{\sin b}{1 + \cos b} \cot \varphi,$$

$$\cos (C - x) = \frac{1 - \cos a}{\sin a} \cot \varphi; \text{ therefore } \frac{\cos (C - x)}{\cos x};$$

$$\text{or, } \cos C + \sin C \tan x = \frac{(1 + \cos b)(1 - \cos a)}{\sin a \sin b}.$$

Substituting, in this equation, the values of cos C and sin C, in terms of the sides a, b, c, and putting for the sake of abridgment, $M = \sqrt{(1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c)}$, we have $\tan x = \frac{1 + \cos b - \cos c - \cos a}{M}$, which determines the angle ACI. From the isosceles triangles ACI, ABI, BCI, we have $ACI = \frac{1}{2}(C + A - B)$; and, in the same manner, $BCI = \frac{1}{2}(B + C - A)$; $BAI = \frac{1}{2}(A + B - C)$.

From which results these remarkable formulæ,

$$\tan \tfrac{1}{2}(A + C - B) = \frac{1 + \cos b - \cos a - \cos c}{M}$$

$$\tan \tfrac{1}{2}(B+C-A) = \frac{1+\cos a-\cos b-\cos c}{M}$$

$$\tan \tfrac{1}{2}(A+B-C) = \frac{1+\cos c-\cos a-\cos b}{M}.$$

To which we may add that which gives $\cot \frac{1}{2} S$, and which can be put under the form

$$\tan \tfrac{1}{2}(A+B+C) = \frac{-1-\cos a-\cos b-\cos c}{M}.$$

From the value of the tangent of x, already found, we have

$$1+\tan^2 x = \frac{1}{\cos^2 x} = \frac{2(1+\cos b)(1-\cos c)(1-\cos a)}{M^2} =$$

$$\frac{16\cos^2 \frac{1}{2} b \sin^2 \frac{1}{2} c \sin^2 \frac{1}{2} a}{M^2};$$

$$\therefore \frac{1}{\cos x} = \frac{4\cos \frac{1}{2} b \sin \frac{1}{2} c \sin \frac{1}{2} a}{M}, \text{ but from the equation}$$

$$\cos x = \frac{1-\cos b}{\sin b}\cot \varphi = \tan \tfrac{1}{2} b \cot \varphi, \text{ we find}$$

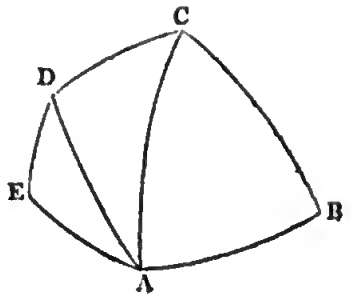
$$\tan \varphi = \frac{\tan \frac{1}{2} b}{\cos x}; \quad \therefore \tan \varphi = \frac{4 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{M}$$

$$= \frac{2 \sin \frac{1}{2} a \sin \frac{1}{2} b \sin \frac{1}{2} c}{\sqrt{\left(\sin \frac{a+b+c}{2} \sin \frac{a+b-c}{2} \sin \frac{a+c-b}{2} \sin \frac{b+c-a}{2}\right)}}.$$

49. The surface of a spherical polygon is measured by the sum of the angles, minus the product of two right angles, and the number of sides of the polygon, minus 2.

From A draw the arcs A C and A D, the angles of the polygon; it will then be divided into as many triangles, minus two, as the figure has sides; but the surface of each triangle is measured by the sum of the angles, minus two right angles; and it is clear, that the sum of all the angles of the triangles is equal to the sum of all the angles of the polygon.

Therefore the surface of the polygon is equal to the sum of the angles diminished by as many times two right angles as the figure has sides, minus two.



Thus if S = sum of the angles of a spherical polygon, n = the number of its sides, then the surface of the polygon is $S - n\pi + 2\pi = S - 2(n-2)$ or $S - 2n + 4$, when the right angle is taken equal to unity.

POLYHEDRONS.

50. If S be the number of solid angles of a polyhedron, H the number of faces, A the number of its edges, then

$$S + H = A + 2.$$

Take a point within the polyhedron, and from which draw lines to all the angular points of the polyhedron; imagine from this point, as a centre, we describe a spherical surface which meets all these lines in as many points, then join these points by arcs of great circles, in such a manner as to form, upon the surface of the sphere, the same number of polygons as there are faces of the polyhedron.

Let ABCDE be one of these polygons, and n the number of its sides, its surface by the last article will be $S - 2n + 4$; S being the sum of the angles A, B, C, D, E. Similarly if we find the value of each of the other spherical polygons, and add them all together, we conclude that their sum or the surface of the sphere which is represented by 8, is equal to the sum of all the angles of the polygons, less twice the number of their sides, plus four times the number of faces.

Now as all the angles that meet at the same point, A, is equal to four right angles, the sum of all the angles of the polygons is equal to four times the number of solid angles, it is therefore equal to 4S. Then double the number of sides AB, BC, CD, &c., is equal to four times the number of edges, or equal to 4A, since the same edge serves for two faces;

$$\therefore 8 = 4S - 4A + 4H; \text{ or}$$
$$2 = S - A + H; \text{ or } S + H = A + 2.$$

[See *Legendre's Geometry*, pp. 228, 229.]

Cor. It follows that the sum of all the plane angles which form the solid angles of a regular polyhedron, is equal to as many times four right angles as there are units in $S - 2$, S being the number of solid angles of the polyhedron.

The plane angles = sum of all the interior angles of each face, which Prop. 32, of the 1st book of Euclid

$$= H(n-2).\pi$$
$$= 2(A - H)\pi \text{ (since } nH = 2A)$$
$$= (S-2)2\pi \text{ (since } A - H = S - 2).$$

51. There can be only five regular polyhedrons.

Since every face has n plane angles, the number of plane angles which compose all the solid angles $= nH = Sm = 2A$, and by the last article $S + H = A + 2$;

$$\therefore H = \frac{m}{n}S, \text{ and } A = \frac{mS}{2}; \text{ substituting these,}$$

$$S + \frac{m}{n}S = \frac{mS}{2} + 2,$$

$$2nS + 2mS = mnS + 4n,$$

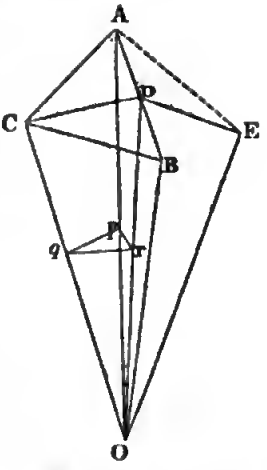
$$2nS + 2mS - mnS = 4n,$$

$$S\{2(n+m) - mn\} = 4n,$$

$$S = \frac{4n}{2(n+m) - mn}.$$

Now this must be a positive whole number, and in order that it may be so $2(m+n)$ must be greater than mn; and, therefore, $\frac{1}{m} + \frac{1}{n} > \frac{1}{2}$, or $\frac{1}{n} > \frac{1}{2} - \frac{1}{m}$; but m cannot be less than 3, therefore, $\frac{1}{n}$ cannot be so small as $\frac{1}{2} - \frac{1}{3}$, or $\frac{1}{6}$, consequently, since n must be an integer and cannot be less than 3, it can only be 3, 4, or 5. In the same manner m cannot be less than 3, therefore the values of m can only be 3, 4, or 5.

52. To find the inclination of two adjacent faces of a polyhedron to each other.



Let AB be the edge common to the two adjacent faces of a polyhedron, C and E the centres of the faces. Draw CO, EO, perpendicular to the faces meeting each other in O; and CD and ED, perpendicular to AB, the intersection of the planes ABC, ABE, then the angle CDE is the required inclination.

Let n be the number of sides in each face, m the number of plane angles in each solid angle if from the centre O, and radius equal to unity, describe a spherical triangle meeting the lines OA, OC, OD in p, q, r, we shall have spherical triangle pqr, in which we have the angle r a right angle, the angle $p = \frac{\pi}{m}$, and the angle $q = \frac{\pi}{n}$;

$$\text{and by right-angled triangles, } \cos qr = \frac{\cos p}{\sin q};$$

but $\cos qr = \cos \text{COD} = \sin \text{CDO} = \sin \frac{1}{2}\text{C}$, C being the angle CDE; then

$$\sin \tfrac{1}{2}\text{C} = \frac{\cos \frac{\pi}{m}}{\sin \frac{\pi}{n}}.$$

This equation is general, and applies successively to the five polyhedrons, by substituting the values of m and n in each case.

Tetrahedron $m = 3, n = 3$. Hexahedron $m = 3, n = 4$.

Octahedron $m = 4, n = 3$. Dodecahedron $m = 3, n = 5$.

Icosahedron $m = 5, n = 3$.

From the triangle pqr from which we have deduced the inclination of the two adjacent faces, we have

$$\cos pq = \cot p \cot q; \text{ or } \frac{\text{CO}}{\text{OA}} = \cot \frac{\pi}{m} \cot \frac{\pi}{n};$$

therefore, if we call R the radius of the sphere which circumscribes the polyhedron, and r the radius of the sphere inscribed in it, we shall have $\frac{R}{r} = \tan\frac{\pi}{m}\tan\frac{\pi}{n}$;

and, by making the side AB $= a$, we have $CA = \frac{\frac{1}{2}a}{\sin\frac{\pi}{n}}$;

$$\text{and, consequently, } R^2 = r^2 + \frac{\frac{1}{4}a^2}{\sin^2\frac{\pi}{n}}.$$

These two equations give for each polyhedron, the values of the radii R and r for the circumscribed and inscribed sphere. We have, supposing C known,

$$r = \tfrac{1}{2}a\cot\frac{\pi}{n}\tan\tfrac{1}{2}C \text{ and } R = \tfrac{1}{2}\tan\frac{\pi}{m}\tan\tfrac{1}{2}C.$$

In the dodecahedron and icosahedron, $\frac{R}{r}$ has the same value for both; viz., $\tan\frac{\pi}{3}\tan\frac{\pi}{5}$. Therefore, if R be the same for both, r will also be the same; that is to say, if these two solids are inscribed in the same sphere, they will also circumscribe the same sphere, and *vice versâ*.

The same property holds with regard to the hexahedron and octahedron, since the value of $\frac{R}{r}$ is the same for one as the other; viz., $\tan\frac{\pi}{3}\tan\frac{\pi}{4}$.

53. To find the inclination of two adjacent faces in the five regular polyhedrons.*

From the equation $\sin\frac{1}{2}C = \frac{\cos\frac{\pi}{m}}{\sin\frac{\pi}{n}}$, taking the tetrahedron

* Legendre, at page 312 of his Geometry, finds the inclination from the equation $\cos C = \frac{\cos c - \cos a\cos b}{\sin a\sin b}$; see equation 3, page 5.

where $m = 3$ and $n = 3$,

$$\sin \tfrac{1}{2} C = \frac{\cos 60^\circ}{\sin 60^\circ} = \frac{1}{\sqrt{3}}; \quad \therefore \cos C = \tfrac{1}{3}.$$

In the hexahedron $m = 3$ and $n = 4$,

$$\sin \tfrac{1}{2} C = \frac{\cos 60}{\sin 45} = \frac{1}{\sqrt{2}}, \text{ and } \cos C = 0; \quad \therefore C = 90.$$

In the octahedron $m = 4$ and $n = 3$,

$$\sin \tfrac{1}{2} C = \frac{\cos 45}{\sin 60} = \sqrt{\frac{2}{3}}, \text{ and } \cos C = -\tfrac{1}{3}.$$

In the dodecahedron $m = 3$, $n = 5$,

$$\sin \tfrac{1}{2} C = \frac{\cos 60}{\sin 36} = \frac{2}{\sqrt{10 - 2\sqrt{5}}}, \text{ and } \cos C = \frac{1 - \sqrt{5}}{5 - \sqrt{5}}.$$

In the icosahedron $m = 5$, $n = 3$,

$$\sin \tfrac{1}{2} C = \frac{\cos 36}{\sin 60} = \frac{1 + \sqrt{5}}{2\sqrt{5}}, \text{ and } \cos C = -\frac{\sqrt{5}}{3}.$$

54. To find the solid content of a regular polyhedron.

The area of each face $= \frac{n}{4} a^2 \cot \frac{\pi}{n}$; hence the area of the surface of the polyhedron $= H \cdot \frac{n}{4} a^2 \cot \frac{\pi}{n}$, and the solid content $= \frac{\text{area of the surface} \times \text{by the altitude}}{3} = \frac{1}{3}$ of the area of the surface $\times$ by radius of inscribed sphere

$$= \tfrac{1}{3} \times H \cdot \frac{n}{4} a^2 \cot \frac{\pi}{n} \times r = \frac{nra^2 H}{12} \cot \frac{\pi}{n} \ldots\ldots (1)$$

$$\left(\text{or since } r = \tfrac{1}{2} a \cot \frac{\pi}{n} \cdot \tan \tfrac{1}{2} C \right)$$

$$= \frac{na^3 H}{12} \cot^2 \frac{\pi}{n} \tan \tfrac{1}{2} C \ldots\ldots\ldots\ldots (2)$$

From either of these equations we can find the solid content.

We shall here use the first. Taking the equations

$$\frac{R}{r} = \tan\frac{\pi}{m}\tan\frac{\pi}{n}, \text{ and } R^2 - r^2 = \frac{a^2}{4\sin^2\frac{\pi}{n}},$$

we can find r and R.

In the tetrahedron $m = 3$, $n = 3$;

$$\therefore \frac{R}{r} = \tan 60 \,.\, \tan 60 = \sqrt{3} \times \sqrt{3} = 3; \quad \therefore R = 3r.$$

$$R^2 - r^2 = \frac{a^2}{4\sin^2 60} = \frac{a^2}{3}, \text{ but } R = 3r.$$

$$\therefore (3r)^2 - r^2 = \frac{a^2}{3} \text{ or } 8r^2 = \frac{a^2}{3};$$

$$\therefore r^2 = \frac{a^2}{24}, \; r = \frac{a}{2\sqrt{6}}. \quad \text{Also } \frac{R}{3} = r$$

$$\therefore R^2 - \frac{R^2}{9} = \frac{a^2}{3}. \qquad R = \frac{3a}{2\sqrt{6}}.$$

In the hexahedron $m = 3$, $n = 4$,

$$\frac{R}{r} = \tan 60 \tan 45 = \sqrt{3}; \; r = \frac{a}{2}; \quad \therefore R = \frac{a\sqrt{3}}{2}.$$

In the octahedron, $m = 4$, and $n = 3$, and $r = \frac{a}{\sqrt{6}}$; $R = \frac{a}{\sqrt{2}}$.

In the dodecahedron, $m = 3$, $n = 5$,

$$r = \frac{a}{20}\sqrt{250 + 110\sqrt{5}}; \; R = \frac{a}{4}(\sqrt{15} + \sqrt{3}).$$

In the icosahedron, $m = 5$, $n = 3$,

$$r = \frac{a}{12}\sqrt{42 + 18\sqrt{5}}; \; R = \frac{a}{4}\sqrt{10 + 2\sqrt{5}}.$$

These being substituted in equation (1), page 65, we find the solidity of each of the five polyhedrons.

For the tetrahedron the solidity is $\frac{a^3}{12}\sqrt{2}$.

For the hexahedron ..... a^3.

For the octahedron..... $\frac{a^3}{3}\sqrt{2}$.

For the dodecahedron ..... $\frac{a^3}{4}\sqrt{470 + 210\sqrt{5}}$.

For the icosahedron ..... $\frac{5a^3}{12}\sqrt{14 + 6\sqrt{5}}$.

Examples.

1. In the oblique-angled spherical triangle ABC. Given the side AB 73° 13′, the side BC 62° 42′, the side AC 119° 5′, required the angles.

Ans. $\begin{cases} A = 44° 18'. \\ B = 136° 40'. \\ C = 48° 48'. \end{cases}$

2. The latitudes and longitudes of three places on the earth's surface, suppose London, Moscow, Constantinople, being given as below: required the latitude and longitude of that place which is equidistant from the former three?

The latitude of London is 51° 30′, the latitude and longitude of Moscow 55° 45′, and 38°, and those of Constantinople 41° 30′ and 29° 15′ respectively.

3. Given the latitude of three places, Moscow 55° 30′, Vienna 48° 12′, Gibraltar 35° 30′, all lying directly in the same arc of a great circle. The difference of longitude between Vienna, (situated in the middle,) and Moscow, easterly, is equal to that between Vienna and Gibraltar, westerly. It is required to find the true bearing and distance of each place from the other, and the difference of longitude, according to the convexity of the globe.

4. Four given equal spheres being placed in close contact with each other, it is required to find the volume of the space inclosed between them and the three triangular planes through each three centres.

5. A point P being taken in the surface of a sphere, let α, β denote its spherical distances from two given points; then if $m \cos \alpha + n \cos \beta =$ a constant quantity, m and n being any given numbers, the locus of P will be a circle.

6. The base of a spherical triangle is given, and the sum of the cosines of the angles at the base, to trace the locus of its vertex.

7. The sides of a spherical triangle are produced to meet again in three more points, thus forming, with the original, four spherical triangles, which constitute Davies's "Associated Triangles;" (12th edit., Hutton's Course, vol. ii. p. 41,) $r\, r_1\, r_2\, r_3$ are the radii of the inscribed, and $R\, R_1\, R_2\, R_3$ the radii of the circumscribed circles. Prove that

$$\tan^2 R + \tan^2 R_1 + \tan^2 R_2 + \tan^2 R_3 = \cot^2 r + \cot^2 r_1 + \cot^2 r_2 + \cot^2 r_3.$$

8. A person engages to travel from London to Constantinople, and to touch the equator in his journey, required the point of contact, and the length of his track, admitting it to be the shortest possible, and the earth a sphere.

9. The angular points of two triangular pyramids being respectively situated on four converging lines in space, let the corresponding faces be produced to meet; then will the four lines of section be all situated in the same plane.

10. Given the longitudes of two places, 6° 49′, and 54° 35′, their respective latitudes 48° 23′ 14″, and 4° 56′ 15″; find their distance, the longitudes being both west, and their latitudes both north.

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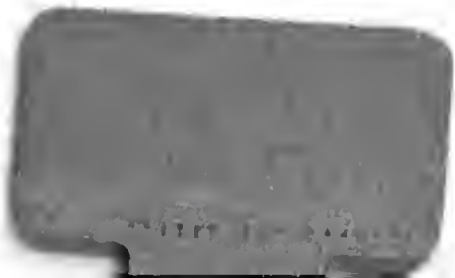

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